Estimation and Inference for Synthetic Control Methods with Spillover Effects[*]

Jianfei Cao[†]   Connor Dowd[‡]

February 21, 2019

Abstract

The synthetic control method is often used in treatment effect estimation with panel data where only a few units are treated and a small number of post-treatment periods are available. Current estimation and inference procedures for synthetic control methods do not allow for the existence of spillover effects, which are plausible in many applications. In this paper, we consider estimation and inference for synthetic control methods, allowing for spillover effects. We propose estimators for both direct treatment effects and spillover effects and show they are asymptotically unbiased. In addition, we propose an inferential procedure and show it is asymptotically unbiased. Our estimation and inference procedure applies to cases with multiple treated units or periods, and where the underlying factor model is either stationary or cointegrated. In simulations, we confirm that the presence of spillovers renders current methods biased and have distorted sizes, whereas our methods yield properly sized tests and retain reasonable power. We apply our method to a classic empirical example that investigates the effect of California's tobacco control program as in Abadie et al. (2010) and find evidence of spillovers.

1   Introduction

The synthetic control method is often used in treatment effect estimation with panel data where only a few units are treated and a small number of post-treatment periods are available. Current estimation and inference procedures for synthetic control methods do not allow for the existence of spillover effects, which are plausible in many applications. This paper alleviates these concerns by providing asymptotically

---

[†]University of Chicago Booth School of Business, jcao0@chicagobooth.edu.

[‡]University of Chicago Booth School of Business, cdowd@chicagobooth.edu.

unbiased estimators and inference in the presence of spillovers. Our results extend to scenarios with multiple treated units and periods, as cases with both stationary and cointegrated factor models.

The synthetic control method (SCM) has gained popularity in empirical studies since its introduction in Abadie and Gardeazabal (2003). When we observe panel data with only a few treated units and post-treatment periods, the SCM can estimate treatment effects. This setting is common in program evaluation, where we often consider state-level policies and have state-level aggregate data. The SCM models the relationship between the treated and untreated units using pre-treatment data. Then the SCM uses the post-treatment data from untreated units to predict the counter-factual values of the treated unit. This process gives us the synthetic control, while the difference between the outcome and predicted counter-factual outcome is the treatment effect estimate. The SCM exploits the pre-treatment data to form better counter-factual values, and so it is often favored over other program evaluation methods such as difference-in-differences. Moreover, while difference-in-differences requires parallel trends between the treatment unit and averaged control units, the SCM works in the much more general case of a factor model. See Abadie and Cattaneo (2018) for review and comparison of econometric methods used in program evaluation.

However, the SCM and its variants assume explicitly or implicitly that untreated units are not affected by the treatment. That is, they rely on the Stable Unit Treatment Value Assumption (SUTVA). This dependence is natural since the SCM uses post-treatment control units to predict the counter-factual values of the treated units, however in context, this is not always realistic. In our empirical example in Section 6, when California imposes a cigarette tax, SUTVA implies (among other things) that nobody decides to shift their cigarette purchases to Nevada.

Under SUTVA and a few other regularity conditions within a factor model, treatment effect estimators using the SCM are shown to be inconsistent but asymptotically unbiased by Ferman and Pinto (2016). Unfortunately, in the presence of a spillover effect, this estimator can be severely biased. Intuitively, the reason is that post-treatment controls are contaminated by the spillover effect, resulting in a biased estimator of the counter-factual value of the treated unit in post-treatment periods, which implies a biased treatment effect estimate. Contamination inducing bias is a standard problem in program evaluation, even within difference-in-differences and RCTs. This problem is worse for the SCM. If by chance the spillover is concentrated in control units that the synthetic control method puts significant weight on, the bias will be substantially worse than in difference-in-differences. Moreover, it is possible the spillovers propagate along the same channels as the underlying factor model, which would mean that the SCM may actively select for units which will induce bias. In our simulation section, we will explore this bias in more depth.

The goal of this paper is to relax the SUTVA condition and to perform estimation and testing. We consider a factor model as the data generating process. To facilitate estimation, we assume that the

treatment effect and the spillover effects are linear in some underlying parameters. We give examples where this assumption can be weak. For each unit of observation, we estimate a model between it and all the other units, using the SCM with pre-treatment data. Thanks to the linear spillover structure, we obtain asymptotically unbiased estimators for the treatment and spillover effects. We also characterize the asymptotic distribution of the estimator.

In addition, we propose an inferential procedure based on Andrews (2003)'s end-of-sample instability test, or P-test. We first generalize the P-test to the synthetic control method without spillover effects and then generalize it further to incorporate cases with spillover effects. Similar to the P-test, our testing procedures use the idea of approximating the null distribution of the statistic using pre-treatment data.

Our estimation and testing procedure can handle both stationary and cointegrated factor models. Furthermore, we consider extensions where treatment applies to multiple units or periods.

We examine an empirical example from Abadie et al. (2010). In 1989 California implemented a cigarette tax. Abadie et al. (2010) gather data from 38 states starting in 1970 for comparison. They dismiss 12 states for potentially being affected by spillovers or later treatment. Despite this precaution, we find evidence of spillover effects in every year after 1990. Moreover, those spillovers appear to have a substantial impact on the treatment effect estimate in 4 of the first 5 years after treatment.

This paper mainly contributes to three developing literatures. First, it complements the fast-developing literature on synthetic control inference by relaxing SUTVA. Due to its popularity among empirical researchers, many formal results have been developed for statistical inference in similar settings. For example, Conley and Taber (2011) consider hypothesis testing in a similar data structure where only a few units are treated and both pre- and post-treatment periods are short. They consider difference-in-differences, which can be treated as a special case of the SCM, and use control units to form the null distribution of the statistic. In Ferman and Pinto (2017) and Hahn and Shi (2017), similar ideas are used to conduct placebo tests which permute across observed units. Among all, Chernozhukov et al. (2017) is the most related to our work, since they also use outcomes across periods rather than across units like the above citations. Li (2017) proposes a testing procedure that is based on the idea of projection onto convex sets and results in Fang and Santos (2018). However, none of the papers mentioned above allows for the existence of spillover effects. Our methods provide formal statistical results in this setting, without assuming SUTVA. Furthermore, our estimation and testing procedure applies to cases where the underlying factor model is cointegrated, which is of special interest even in cases without spillover effects.

We also contribute to the literature on spillover effects. This fast-growing literature looks into both estimation of treatment effects in the presence of spillover effects, as well as estimation of spillover effects themselves. For example, Vazquez-Bare (2017) consider a framework where observations are grouped into clusters, and spillover effects are allowed within a cluster, but not across clusters. It discusses

estimation of heterogeneous treatment effects as a function of the number of treated units within the same cluster, and spillover effects as a function of whether the unit is treated, and number of treated units within the same cluster. Basse et al. (2017) consider a relatively general form of spillover effects and use randomization test for inference. Also see both Vazquez-Bare (2017) and Basse et al. (2017) for a literature review on spillover effects. However, this literature seldom looks at the panel data setting with only a few treated units and short post-treatment periods. This limitation is in part because we usually do not have enough information about the spillover effects in this particular setting. We overcome this problem by requiring a potentially weak assumption that the spillover structures be pre-specified and follow a pattern that is linear in some underlying parameters. With that specification, we can estimate the spillover effects and perform statistical tests on the spillovers.

Third, our results extend the literature on Andrews (2003)'s end-of-sample instability tests. Andrews (2003) uses data across time periods to approximate the null distribution of the test statistic, and apply this idea to OLS, IV, and GMM. Chernozhukov et al. (2017) propose a permutation method that is more general, but similar in cases where serial correlation matters. We extend this idea to the the SCM case, and further to more complicated cases with spillover effects. As Andrews and Kim (2006) extends Andrews (2003)'s results to the cointegrated cases, we also show that our method is still valid for a cointegrated factor model.

The remainder of this paper is organized as follows. Section 2 introduces a factor model with spillover effects, proposes an estimator of the spillover effects and derives its asymptotic distribution. Section 3 considers the P-test introduced by Andrews (2003) and Andrews and Kim (2006), and explains how it can be applied in our settings, with proofs in the Appendix. Section 4 extends our methods to cases with multiple treated units and/or multiple post-treatment periods. We present Monte Carlo simulation results in Section 5 and in Section 6 we present an empirical example of our method. Section 7 concludes.

2 Identification and Estimation

2.1 Model

We consider the Rubin's potential outcome model with underlying factor structure. To set up the model, we assume for now only one unit is treated, and extend to cases with multiple treated units in Section 4. That is, let

$$y_{i,t} = \begin{cases} y_{i,t}(1), & \text{if } t = T+1, \\ y_{i,t}(0), & \text{otherwise} \end{cases} \tag{1}$$

for $i = 1, \ldots, N$ and $t = 1, \ldots, T+1$. Note that only unit 1 is treated at $t = T+1$. For $i \neq 1$, the potential difference between $y_{i,T+1}(1)$ and $y_{i,T+1}(0)$ implies a deviation from unit i's counterfactual outcome $y_{i,T+1}(0)$ where no unit is treated at time $T+1$. We consider the case where N is fixed and T

goes to infinity. For simplicity, we only consider the case with one post-treatment period, but everything discussed here can be naturally extended to cases with multiple post-treatment periods (see Section 4).

Let $\alpha_i = y_{i,T+1}(1) - y_{i,T+1}(0)$, which can be the treatment effect or spillover effect depending on whether $i = 1$. We are interested in estimating the treatment effect α_1, and a common choice is the synthetic control estimator. Denoting $x_t = (1, y_{1,t}, y_{2,t}, \ldots, y_{N,t})'$, then a synthetic control weight estimator for a given closed convex set $\Lambda \subset \mathbb{R}^{N+1}$ is

$$\hat{\beta}_\Lambda = \arg\min_{\beta \in \Lambda} \sum_{t=1}^{T} (y_{1,t} - x_t'\beta)^2. \tag{2}$$

An estimator of the treatment effect α_1 is given by

$$\hat{\alpha}_1 = y_{1,T+1} - x_{T+1}'\hat{\beta}_\Lambda, \tag{3}$$

i.e. the counter-factual value $y_{1,T+1}(0)$ is approximated by $x_{T+1}'\hat{\beta}_\Lambda$. For this paper we use $\Lambda = \mathbb{R} \times \{0\} \times \Delta_{N-1}$, where $\Delta_{N-1} = \{\theta \in \mathbb{R}^{N-1} : \theta_i \geq 0 \text{ for each } i, \sum_{i=1}^{N-1} \theta_i = 1\}$ is a $(N-1)$-dimensional simplex. That is, we do not restrict the intercept but require other coefficients to be positive and sum up to one. Note that the choice of Λ depends on what a researcher believes and is not essential to our analysis. See Doudchenko and Imbens (2016) for a discussion of choice of the restriction set.[1]

We follow Ferman and Pinto (2016) and consider a factor model such that for $i = 1, \ldots, N$ and $t = 1, \ldots, T+1$,

$$y_{i,t}(0) = \eta_t + \lambda_t'\mu_i + \epsilon_{i,t}, \tag{4}$$

where λ_t is F-dimensional common factors, and $\epsilon_{i,t}$ is noise that is uncorrelated with λ_t. For notation simplicity, we write $Y_t(0) = (y_{1,t}(0), \ldots, y_{n,t}(0))'$, $Y_t = (y_{1,t}, \ldots, y_{n,t})'$, and $\epsilon_t = (\epsilon_{1,t}, \ldots, \epsilon_{n,t})'$.

Assume that the spillover effect is a linear transformation of some unknown parameter $\gamma \in \mathbb{R}^k$, i.e. $\alpha = A\gamma$. Typically, γ has less dimensions than α does. Here are some examples that fit in this framework.

Example 1. *Assume a subset of control units, but not all of them, are equally affected by the spillover effects, i.e.*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \ \gamma = \begin{bmatrix} \alpha_1 \\ b \end{bmatrix}.$$

[1] Other choices of Λ include $\{0\} \times \{0\} \times \Delta_{N-1}$ (the original synthetic control method in Abadie and Gardeazabal (2003) and Abadie et al. (2010)), and $\mathbb{R} \times \{0\} \times \mathbb{R}_+^{N-1}$ (modified synthetic control in Li (2017)).

Example 2. *Assume the spillover effect shrinks as the geometric distance goes up. For $i = 2, \ldots, N$, $\alpha_i = b \exp(-d_i)$ where d_i is the distance between unit 1 and unit i and b is some unknown parameter of interest. Then, we have*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-d_2) \\ \vdots & \vdots \\ 0 & \exp(-d_N) \end{bmatrix}, \ \gamma = \begin{bmatrix} \alpha_1 \\ b \end{bmatrix}.$$

Example 3. *Assume the spillover effect takes place at some known locations, but not at other locations, while the sizes of spillover effects are allowed to vary across those units. For example, assume there are spillovers at locations whose distance to unit 1 is less than $\bar{d}$. Then, the treatment and spillover effect vector can also be represented by $A\gamma$. WLOG order the units by increasing distance from unit 1, and let p the number of units experiencing spillovers. Then*

$$A = \begin{bmatrix} 1 & 0_{1 \times p} \\ 0_{p \times 1} & I_p \\ 0_{(N-p-1) \times 1} & 0_{(N-p-1) \times p} \end{bmatrix}, \ \gamma = \begin{bmatrix} \alpha_1 \\ \alpha_{k_1} \\ \vdots \\ \alpha_{k_p} \end{bmatrix}.$$

Thus the units indexed $2, \ldots, (p+1)$ each experience their own size spillover effect.

2.2 Identification

In order to back out the spillover effects, we re-formulate the model in the following way. For each i and t, we write $y_{i,t}(0)$ as a constant plus a weighted average of other outcomes at time t. That is, let

$$y_{i,t}(0) = a_i + Y_t(0)'b_i + u_{i,t}, \tag{5}$$

where the i-th entry of b_i is zero, and a_i and b_i are the probability limit of the synthetic control weight estimator and $u_{i,t}$ is defined by this equation. Namely, let

$$\begin{bmatrix} \hat{a}_i \\ \hat{b}_i \end{bmatrix} = \arg\min_{(a,b) \in W_i} \sum_{t=1}^{T} (y_{i,t} - a - Y_t b')^2, \tag{6}$$

where $W_i = \{\beta = (\beta_0, \beta_1, \ldots, \beta_N)' \in \mathbb{R} \times \mathbb{R}_+^N : \beta_i = 0, \sum_{j=1}^{N} \beta_j = 1\}$. Then, let

$$a_i = \text{plim } \hat{a}_i, \ b_i = \text{plim } \hat{b}_i, \tag{7}$$

6

and we only consider cases where they are well-defined (see Lemma 1 for examples where $\hat{a}_i$ and $\hat{b}_i$ converge). Stacking equation (5) for all i's gives

$$Y_t(0) = a + BY_t(0) + u_t, \tag{8}$$

where $a = (a_1, \ldots, a_N)'$, i-th row of B is b_i' and $u_t = (u_{1,t}, \ldots, u_{N,t})'$. For $t = T+1$, this becomes

$$(I - B)(Y_{T+1} - \alpha) - a = u_{T+1}, \tag{9}$$

where $Y_{T+1} = (y_{1,T+1}, \ldots, y_{N,T+1})'$. We want to use this equation to estimate the spillover effect.

Note that (5) is not obtained by solving for λ_t using (4). In general, a_i and b_i will the probability limit of regressing $y_{i,t}$ on other outcomes using synthetic control methods, which typically does not coincide with the weights that reconstruct the factor loadings (Ferman and Pinto (2016)).

Defining $M = (I - B)'(I - B)$, we can introduce the following identification assumption:

Condition ID. $A'MA$ *is non-singular.*

First note Condition ID is testable in principle. We can consistently estimate B so the data informs us of the validity of this assumption. To understand this assumption better, we replace α by $A\gamma$ in Equation (9) and have

$$(I - B)A\gamma = (I - B)Y_{T+1} - a - u_{T+1}. \tag{10}$$

Equation (10) is the key to identification. Under mild assumptions, a and B are identified from the model and learned by the synthetic control method. We do not observe u_{T+1}, but the distribution of u_{T+1} can be learned using pre-treatment data under stationarity of $\{u_t\}_{t \geq 1}$. Therefore, if $A'MA$ is non-singular, or equivalently, $(I - B)A$ has full rank, we can form an estimator of γ whose limiting distribution is identified by multiplying both sides of Equation (10) by $(A'MA)^{-1}A'(I - B)'$. Note that the notion of identification is different from the usual sense, we do not identify γ or α. This is because we have only one observation of the outcome in post-treatment periods.

We illustrate Condition ID in the following toy example.

Example 4. *Assume there are 3 units in total, where unit 1 is treated. Let the synthetic control weight matrix B be*

$$B = \begin{bmatrix} 0 & w_1 & 1 - w_1 \\ w_2 & 0 & 1 - w_2 \\ w_3 & 1 - w_3 & 0 \end{bmatrix}.$$

Suppose the researcher first assumes unit 2 and 3 are equally exposed to the spillover effects. That is,

they assume

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \ \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \ and \ \alpha = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_2 \end{bmatrix}.$$

Then, Condition ID does not hold, because

$$(I - B)A_1 = \begin{bmatrix} 1 & -1 \\ -w_2 & w_2 \\ -w_3 & w_3 \end{bmatrix}.$$

If they instead assumes only one of the controls is exposed to the spillover effects, Condition ID is satisfied in general. In this case,

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \ and \ \alpha = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{bmatrix},$$

and

$$(I - B)A_2 = \begin{bmatrix} 1 & -w_1 \\ -w_2 & 1 \\ -w_3 & w_3 - 1 \end{bmatrix}.$$

The only case where $(I - B)A_2$ does not has full rank is when $(w_1, w_2, w_3) = (1, 1, 1/2)$.

This applies to more general settings. That is, if all controls are equally hit by the spillover effects, then $(I - B)A$ does not have full rank and we lose identification. Allowing a few units to exempt from the spillover effects makes $(I - B)A$ have full rank in general.

A more interesting case is Example 3, where we only restrict the range of spillover effects and allow the levels to vary. In this case, $(I - B)A$ can be obtained by eliminating columns that correspond to units that are neither treated nor exposed to spillover effects. Again, as long as at least one control is not exposed to the spillover effects, $(I - B)A$ has full rank in general. This assumption is more convincing if a moderate number of columns are eliminated from $(I - B)$, i.e. only a few units are exposed to the spillover effects.

2.3 Estimation

We focus on two sets of conditions in our discussion.

Condition ST (model with stationary common factors)**.** *Assume $\{(\eta_t, \lambda_t, \epsilon_t)\}_{t \geq 1}$ is stationary, ergodic for the first and second moments, and has finite $(2+\delta)$-moment for some $\delta > 0$. Assume $cov[Y_t(0)] = \Omega_y$ is positive definite.*

Remarks: 1. We show in the proof of Lemma 1 that in this case

$$b_i = \underset{w \in W^{(i)}}{\arg\min} \; (w - e_i)'\Omega_y(w - e_i), \tag{11}$$

$$a_i = E[y_{i,1}(0) - Y_1(0)'b_i], \tag{12}$$

where e_i is a unit vector with one at the i-th entry and zeros everywhere else, and $W^{(i)} = \{(w_1, \ldots, w_N) \in \mathbb{R}^N_+ : w_i = 0, \sum_{j \neq i} w_j = 1\}$. Note that in general b_i does not recover the factor structure, because $\mu_i \neq (\mu_1, \ldots, \mu_N)b_i$ in general.

2. We do not impose any restriction on the factor loadings $\{\mu_i\}_{i=1}^N$ except for Ω_y being positive definite. In the stationary case, the key for the treatment estimator to be asymptotically unbiased and the test proposed below to be valid is to include an intercept in the optimization problem (6).

Condition CO (model with cointegrated $\mathcal{I}(1)$ common factors). *Rewrite Equation (4) as*

$$y_{i,t}(0) = (\lambda_t^1)'\mu_i^1 + (\lambda_t^0)'\mu_i^0 + \epsilon_{i,t}, \tag{13}$$

and η_t can be either in λ_t^1 or λ_t^0. Assume $\{(\lambda_t^0, \epsilon_t)\}_{t \geq 1}$ is stationary, ergodic for the first and second moments, and has finite 4-th moment. Without loss of generality, $E[\epsilon_{i,t}] = 0$. Assume $\{\lambda_t^1\}_{t \geq 1}$ is $\mathcal{I}(1)$. Further assume for each i, $y_{i,t}(0)$ is such that weak convergence holds for $T^{-1/2}\sum_{t=1}^{[rT]} y_{i,t}(0) \Rightarrow \nu_i(r)$, where $\Rightarrow$ is weak convergence and process $\nu_i(r)$ is defined on $[0,1]$ and has bounded continuous sample path almost surely. For each i, let $W^{(i)} = \{(w_1, \ldots, w_N) \in \mathbb{R}^N_+ : w_i = 0, \sum_{j \neq i} w_j = 1\}$. Assume for each i, there exists $w^{(i)} \in W^{(i)}$ such that $\mu_i^1 = \sum_{j=1}^N w_j^{(i)}\mu_j^1$. That is, $(w^{(i)} - e_i)$ is a cointegrating vector for $Y_t(0)$, where e_i is a unit vector with i-th entry being one and zeros everywhere else.

We form estimators for (a, B) using synthetic control methods as in (6). We do that for each $i = 1, \ldots, N$, as if each i is the treated unit and other units are controls. Then, the estimators for a and B are $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N)'$ and $\hat{B} = (\hat{b}_1, \ldots, \hat{b}_N)'$ respectively. Let $\hat{M} = (I - \hat{B})'(I - \hat{B})$ be an estimator for M. Let an estimator of γ be such that

$$\hat{\gamma} = \underset{g \in \mathbb{R}^k}{\arg\min} \|(I - \hat{B})(Y_{T+1} - Ag) - \hat{a}\|$$

$$= (A'\hat{M}A)^{-1}A'(I - \hat{B})'((I - \hat{B})Y_{T+1} - \hat{a}). \tag{14}$$

Note that the FOC implies

$$A'(I - B)'u_{T+1} = 0, \tag{15}$$

i.e. it requires that some weighted sum of the residuals to be zero. Under that condition, the treatment and spillover effect vector α can be estimated by $\hat{\alpha} = A\hat{\gamma}$.

Assumption 1. (a) $\{u_t\}_{t \geq 1}$ *is stationary, and has mean zero.*

(b) $\|\hat{a} - a\| = o_p(1)$, $\|\hat{B} - B\| = o_p(1)$

(c) $\|(\hat{B} - B)Y_{T+1}(0)\| = o_p(1)$.

(d) $A'MA$ *is non-singular.*

Note that Part (c) excludes polynomial time trends.

Lemma 1. *Under Condition ID, either Condition ST or Condition CO implies Assumption 1.*

Theorem 1. *Suppose Assumption 1 holds. Then, $\hat{\alpha} - (\alpha + Gu_{T+1}) \to_p 0$ as $T \to \infty$, where $G = A(A'MA)^{-1}A'(I - B)'$. Moreover, $E[Gu_{T+1}] = 0$.*

The structure of the limiting distribution is similar to the case as in Ferman and Pinto (2016), as it is inconsistent but asymptotically unbiased (i.e. that the difference between the estimator and the true value has zero mean). Note that linearity of α in γ is crucial in order to obtain the asymptotic unbiasedness.

2.4 Efficiency

In the spirit of GMM with an efficient weighting matrix, we now form an estimator of α with possibly lower variance. For a positive definite matrix $W \in \mathbb{R}^N$, we minimize $\|W^{1/2}\epsilon_{T+1}\|$ instead of $\|\epsilon_{T+1}\|$. The resulting estimator is

$$\hat{\gamma}_W = \arg\min_{g \in \mathbb{R}^k} \|W^{1/2}((I - \hat{B})(Y_{T+1} - Ag) - \hat{a})\|$$

$$= (A'\hat{M}_W A)^{-1}A'(I - \hat{B})'W((I - \hat{B})Y_{T+1} - \hat{a}), \tag{16}$$

where $\hat{M}_W = (I - \hat{B})'W(I - \hat{B})$. The corresponding estimator for α is $\hat{\alpha}_W = A\hat{\gamma}_W$. Let $\Omega = Cov[u_1]$ and W_T^e be a consistent estimator of Ω^{-1}. Then an estimator of α with lower variance can be achieved by

$$\hat{\alpha}^e = \hat{\alpha}_{W_T^e} \tag{17}$$

Let $M_W = (I - B)'W(I - B)$, $G_W = A(A'M_W A)^{-1}A'(I - B)'W$ for some weighting matrix W, $W^e = \Omega^{-1}$, $M^e = M_{W^e}$, and $G^e = G_{W^e}$. Then, we have the following results.

Proposition 1. *Suppose Assumption 1 holds, W_T is a consistent estimator for W, and W_T^e is a consistent estimator for W^e. Then, $\hat{\alpha}_{W_T} - (\alpha + G_W u_{T+1}) \to_p 0$, and specifically, $\hat{\alpha}^e - (\alpha + G^e u_{T+1}) \to_p 0$, as $T \to \infty$. Moreover, $(Cov[G_W u_{T+1}] - Cov[G^e u_{T+1}])$ is positive semi-definite.*

In practice, we need to estimate Ω, and for that we would need relatively large sample size (large T) to have a good approximation.

3 Inference

In this section, we discuss formal results on inference. At a high level, our test uses pre-treatment data to form the null distribution of a pre-specified post-treatment quantity. Flexibility in defining that quantity leads to a variety of hypotheses. In Section 3.1, we consider the case without spillover effects, and state the assumptions under which Andrews' P test (Andrews (2003)) is valid. In Section 3.2, we generalize P test to cases where spillover effects cannot be ignored.

3.1 Cases without spillover effects

Suppose for now there are no spillover effects, i.e. $\alpha_2 = \cdots = \alpha_N = 0$. We want to test for the existence of treatment effect on unit 1. The null and alternative hypotheses of interest are

$$\begin{cases} H_0 : \alpha_1 = 0, \\ H_1 : \alpha_1 \neq 0. \end{cases} \tag{18}$$

The test procedure we consider here is the end-of-sample instability test (P-test) in Andrews (2003). The usage of Andrews' test in the context of synthetic control methods is mentioned in Ferman and Pinto (2018), where they focus on the difference-in-differences estimator. We formalize this idea and consider both stationary and cointegrated series.

We assume the α_1 is independent of T under H_1. That is, we consider fixed, not local, alternatives, as in Andrews (2003) and Andrews and Kim (2006). Specifically, α_1 does not change as T grows, which facilitates our analysis of the test statistic under H_1.

Now we translate our hypothesis into the linear formulation considered in Andrews (2003). Namely, we have

$$y_t = \begin{cases} a_1 + Y_t' b_1 + u_{1,t}, & \text{for } t = 1, \ldots, T, \\ a_1^* + Y_t' b_1 + u_{1,t}, & \text{for } t = T + 1. \end{cases} \tag{19}$$

A non-zero treatment effect is equivalent to a shift in the intercept a_1 (or equivalently, change of the distribution of $u_{1,t}$, at $t = T + 1$). The null and alternative hypothesis (18) become

$$\begin{cases} H_0 : a_1^* = a_1, \\ H_1 : a_1^* \neq a_1. \end{cases} \tag{20}$$

Let the synthetic control regression residuals be $\hat{u}_{1,t} = y_{1,t} - \hat{a}_1 - Y_t' \hat{b}_1$. The test statistic is defined by

$$P = \hat{u}_{1,T+1}^2. \tag{21}$$

For notational simplicity, let $\hat{\beta}_1 = (\hat{a}_1, \hat{b}'_1)'$ and $x_t = (1, Y'_t)'$. For $\beta \in \mathbb{R}^{N+1}$, define

$$P_t(\beta) = (y_{1,t} - x'_t\beta)^2. \tag{22}$$

Then, $P = (y_{1,T+1} - x'_{T+1}\hat{\beta}_1)^2 = P_{T+1}(\hat{\beta}_1)$. Let P_∞ be a random variable with the same distribution as $P_{T+1}(\beta_1)$ with $\beta_1 = (a_1, b'_1)'$.

Let $P_t = P_t(\hat{\beta}_1^{(t)})$, where $\hat{\beta}_1^{(t)} = \hat{\beta}_1$ for each t.[2] Define

$$\hat{F}_{P,T}(x) = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\{P_t \leq x\}, \tag{23}$$

and let $F_P(x)$ be the distribution function of $P_1(\beta_1)$. Finally, let $\hat{q}_{P,1-\tau} = \inf\{x \in \mathbb{R} : \hat{F}_{P,T}(x) \geq 1 - \tau\}$, and $q_{P,1-\tau}$ be the $(1-\tau)$-quantile of $P_1(\beta_1)$.

Assumption 2. (a) $\{u_t\}_{t\geq 1}$ are stationary, ergodic, and have mean zero.

(b) $E[\|u_t\|] < \infty$.

(c) $\exists$ a non-random sequence of positive definite matrices $\{C_T\}_{T\geq 1}$ such that $\max_{t\leq T+1}\|C_T^{-1}x_t\| = O_p(1)$

(d) $\|C_T(\hat{\beta}_1 - \beta_1)\| = o_p(1)$, and $\max_{t=1,\ldots,T}\|C_T(\hat{\beta}_1^{(t)} - \beta_1)\| = o_p(1)$.

(e) The distribution function of $P_1(\beta_1)$ is continuous and increasing at its $(1-\tau)$-quantile.

Lemma 2. Suppose the distribution function of $P_1(\beta_1)$ is continuous and increasing at its $(1-\tau)$-quantile. Then, either Condition ST or Condition CO implies Assumption 2.

Theorem 2. Suppose Assumption 2 holds. Then, as $T \to \infty$,

(a) $P \to_d P_\infty$ under H_0 and H_1,

(b) $\hat{F}_{P,T}(x) \to_p F_P(x)$ for all x in a neighborhood of $q_{P,1-\tau}$ under H_0 and H_1,

(c) $\hat{q}_{P,1-\tau} \to_p q_{P,1-\tau}$ under H_0 and H_1,

(d) $\Pr(P > \hat{q}_{P,1-\tau}) \to \alpha$ under H_0.

3.2 Cases with spillover effects

Now we allow for non-zero spillover effects. We propose a testing procedure that is based on Andrews' P-test and accounts for the spillover effect. The null and alternative hypotheses we consider are $H_0 : C\alpha = d$ and $H_1 : C\alpha \neq d$, with C and d known. For example, we want to test for the hypothesis that there is no treatment effect at the treated unit (unit 1), then we let $C = (1, 0, 0, \ldots, 0) \in \mathbb{R}^{1\times N}$ and $d = 0$. This effectively makes Section 3.1 a special case of our test (although Theorem 2 has slightly stronger results than Theorem 3 does). If we want to test that there is a spillover, then we can let $C = [0_{(N-1)\times 1}\ I_{N-1}] \in \mathbb{R}^{(N-1)\times N}$ and $d = (0, \ldots, 0)' \in \mathbb{R}^{(N-1)\times 1}$.

---

[2]You can also use leave-one-estimator to construct P_t as in Andrews (2003) and Andrews and Kim (2006). For $t = 1, \ldots, T$, the leave-one-out estimator $\hat{\beta}_1^{(t)}$ is defined by the synthetic control weight estimator using only observations indexed by $s = 1, \ldots, t-1, t+1, \ldots, T$.

The test statistic we consider here is $P = (C\hat{\alpha} - d)'W_T(C\hat{\alpha} - d)$ for some weighting matrix $W_T \to_p W$. Recall $G = A(A'MA)^{-1}A'(I - B)$ and can be consistently estimated by $\hat{G} = A(A'\hat{M}A)^{-1}A'(I - \hat{B})$ if $\hat{B} \to_p B$. By Theorem 1, P is asymptotically equivalent to $u'_{T+1}G'C'WCGu_{T+1}$. To construct critical values, define

$$P_t(\theta) = (Y_t - \theta x_t)'G'C'WCG(Y_t - \theta x_t), \tag{24}$$

and

$$\hat{P}_t(\theta) = (Y_t - \theta x_t)'\hat{G}'C'W_TC\hat{G}(Y_t - \theta x_t), \tag{25}$$

for some $\theta \in \mathbb{R}^{N \times (N+1)}$, $x_t = (1, Y'_t)'$, and $\hat{G} = A(A'\hat{M}A)^{-1}A'(I - \hat{B})'$. Let $\hat{P}_t = \hat{P}_t(\hat{\theta}^{(t)})$, where $\hat{\theta}^{(t)} = \hat{\theta}$ for each t.[3] Let $P_\infty = P_1(\theta_0)$ for $\theta_0 = [a\ B]$. Define

$$\hat{F}_{P,T}(x) = \frac{1}{T}\sum_{j=1}^{T} \mathbb{1}\{\hat{P}_t \leq x\}, \tag{26}$$

and let $F_P(x)$ be the distribution function of P_∞. Finally, let $\hat{q}_{P,1-\tau} = \inf\{x \in \mathbb{R} : \hat{F}_{P,T}(x) \geq 1 - \tau\}$, and $q_{P,1-\tau}$ be the $(1 - \tau)$-quantile of P_∞.

Assumption 3. (a) *Assumption 1 holds.*

(b) $\{u_t\}_{t\geq 1}$ *is ergodic and* $E[\|u_t\|] < \infty$.

(c) *There exists a non-random sequence of positive definite matrices* $\{D_T\}_{T\geq 1}$ *such that* $\max_{t\leq T+1}\|D_T^{-1}x_t\| = O_p(1)$.

(d) $\|(\hat{\theta} - \theta_0)D_T\|_F = o_p(1)$, *and* $\max_{t=1,\ldots,T}\|(\hat{\theta}^{(t)} - \theta_0)D_T\|_F = o_p(1)$, *where* $\|\cdot\|_F$ *is the Frobenius norm.*

(e) *The distribution function of* $P_1(\theta_0)$ *is continuous and increasing at its* $(1 - \tau)$-*quantile.*

(f) $W_T \to_p W$ *as* $T \to \infty$.

Lemma 3. *Suppose the distribution function of* $P_1(\theta_0)$ *is continuous and increasing at its* $(1-\tau)$-*quantile. Then, under Condition ID, Assumption 3 is satisfied if either of these holds:*

(i) *Condition ST with* $W_T = I$ *or* $W_T = (C\hat{G}(T^{-1}\sum_{t=1}^{T}\hat{u}_t\hat{u}'_t)\hat{G}'C')^{-1}$;

(ii) *Condition CO with* $W_T = I$.

Theorem 3. *Suppose Assumption 3 holds. Then, under* H_0, *as* $T \to \infty$,

(a) $P \to_d P_\infty$,

(b) $\hat{F}_{P,T}(x) \to_p F_P(x)$ *for all* x *in a neighborhood of* $q_{P,1-\tau}$,

(c) $\hat{q}_{P,1-\tau} \to_p q_{P,1-\tau}$,

(d) $\Pr(P > \hat{q}_{P,1-\tau}) \to \alpha$.

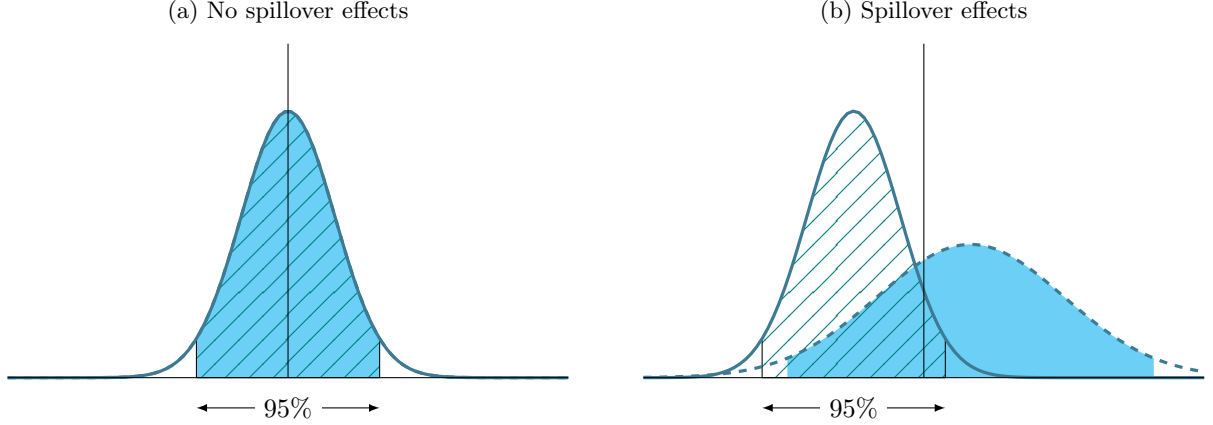| (a) No spillover effects | (b) Spillover effects |

Figure 1: Placebo test. Area with lines is 95% probability region of the error of the treated unit. Filled area is 95% probability region of null distribution formed in placebo test. A test is rejected when the error of the treated units falls outside of the filled area.

3.3 Other testing procedures

When we allow for existence of non-zero spillover effects, the existing testing procedures will have poor performance. Here we intuitively explain what happens to placebo test as in Abadie and Gardeazabal (2003) and Andrews' test as in Andrews (2003) in the presence of spillover effects.

Suppose we want to test for the treatment effect being zero and are not aware of the spillover effects. Placebo test and Andrews' test are similar in the sense that they use data to form the null distribution of $u_{1,T+1}$ in order to perform hypothesis testing. The difference is that the placebo test exploits variations of $\{\hat{u}_{i,T+1}\}_{i=1}^{N}$, while Andrews' test uses variations of $\{\hat{u}_{1,t}\}_{t=1}^{T+1}$.

We look at the placebo test first. When there is no spillover effect, the distribution of $\hat{u}_{1,T+1}$ and distribution of $\{\hat{u}_{i,T+1}\}_{i=2}^{N}$ overlap asymptotically. As shown in Figure 1(b), when there are positive spillover effects, we will underestimate the treatment effect and the density function of $\hat{u}_{1,T+1}$ moves to the left. At the same time, some of the control units shift to the right because of the positive spillovers, so density of $\{\hat{u}_{i,T+1}\}_{i=2}^{N}$ moves to the right and gets wider. In terms of test performance, the shift of $\hat{u}_{1,T+1}$ is offset by the wider density of $\{\hat{u}_{i,T+1}\}_{i=2}^{N}$ (harder to reject H_0), which explains why in Table 3 of Section 5 the empirical sizes of placebo test for $T = 50$ and 200 cases are not too far away from the nominal size 0.05. In essence, the placebo test becomes much more conservative and has low power as shown in Table 4.

Now we consider Andrews' test. When there is no spillover effect, the distribution of $\hat{u}_{1,T+1}$ and distribution of $\{\hat{u}_{1,t}\}_{t=1}^{T}$ overlap asymptotically. As shown in Figure 2(b), when there is positive spillover effect, we underestimate the treatment effect and the density function of $\hat{u}_{1,T+1}$ shifts to the left, while the density of $\{\hat{u}_{1,t}\}_{t=1}^{T}$ doesn't, since they are pre-treatment and the spillover only happens after the

---

[3]Similar to the case without spillover effects, the leave-one-out estimator $\hat{\theta}^{(t)} = [\hat{a}^{(t)}\ \hat{B}^{(t)}]$ is defined by the synthetic control weight estimator using only observations indexed by $s = 1, \ldots, t-1, t+1, \ldots, T$.

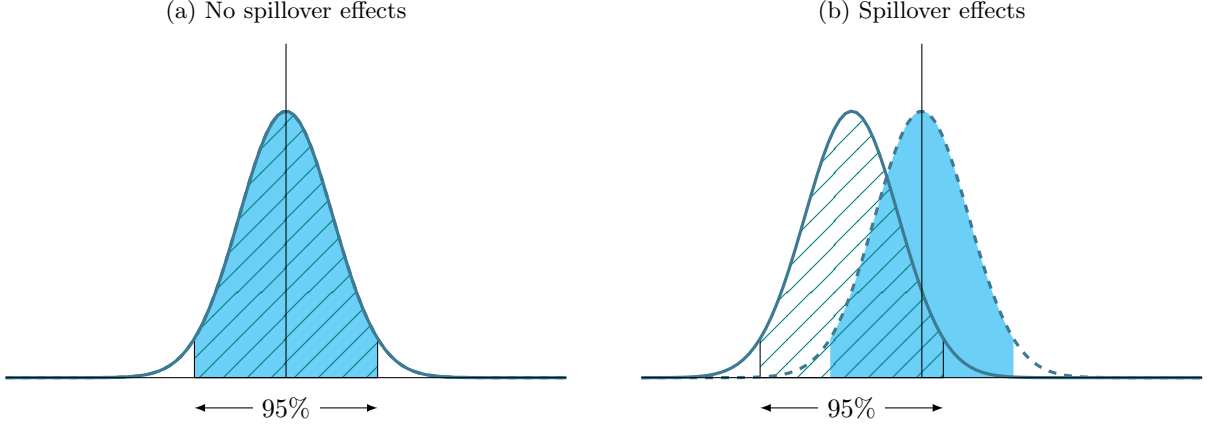(a) No spillover effects   (b) Spillover effects

Figure 2: Andrews' test. Area with lines is 95% probability region of the error of the treated unit. Filled area is 95% probability region of null distribution formed in Andrews' test. A test is rejected when the error of the treated units falls outside of the filled area.

treatment. This results in an invalid test.

4 Extensions

4.1 Multiple treated units

Our method readily extends to cases where multiple units are treated. In our setting, spillover effects are not distinguished from treatment effects, since one can think of spillover as the treatment on the units that are not directly treated. With a corrected specified structure matrix A, we can perform estimation and testing just as previous sections. For example, suppose $N = 4$, unit 1 and unit 2 are treated, unit 3 is affected by spillover effect, and unit 4 is neither treated nor exposed to spillover effect. Then we can specify

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

and the resulting estimator $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)'$ by (14) is such that $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the treatment effect estimator for unit 1 and unit 2, respectively, and $\hat{\gamma}_3$ is the spillover effect estimator for unit 3. Tests can be performed accordingly. If one wants to test for the hypothesis that there are no spillover effects, the null is then $H_0 : C\alpha = d$, where $C = (0, 0, 1, 0)$ and $d = 0$.

4.2 Multiple post-treatment time periods

Suppose now we have observations of $\{y_{i,t}\}$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T + m$. Treatment is received at $t = T + 1$. The model becomes

$$Y_t = \begin{cases} Y_t(0), \text{ if } t \leq T \\ Y_t(0) + \alpha_t, \text{ otherwise.} \end{cases} \tag{28}$$

Note that we do not allow for spillovers in time. That is, the treatment effect or spillover effects cannot affect future selves. For each $t = T + 1, \ldots, T + m$, we need to specify the spillover structure matrix A_t. Then, an estimator of α_t is

$$\hat{\alpha}_t = A_t(A_t' \hat{M} A_t)^{-1} A_t'(I - \hat{B})'((I - \hat{B})Y_t - \hat{a}). \tag{29}$$

That is, we treat $T + s$ period as $T + 1$ and do the same procedure as before. For each $t = T + 1, \ldots, T + m$, we can perform separate tests as introduced in previous sections.

To answer simultaneous questions such as whether there is spillover effect at all, we can extend the P-test discussed above. Consider the null hypothesis $H_0 : C_t \alpha_t = d_t$ for $t = T + 1, \ldots, T + m$. Let $\hat{P}_t$ be constructed as in Section 3.2 for $t = 1, \ldots, T$. For $t = T + 1, \ldots, T + m$, let $\hat{P}_t = (C_t \hat{\alpha}_t - d_t)' W_T (C_t \hat{\alpha}_t - d_t)$. We can now form

$$P^{(t)} = \sum_{s=0}^{m-1} \hat{P}_{t+s} \tag{30}$$

for $t = 1, \ldots, T + 1$. The test statistic is then $P^{(T+1)}$, and we use $\{P^{(t)}\}_{t=1}^T$ to form its null distribution.

5 Simulation

We present Monte Carlo simulation results in this section. For each case considered, we use 1000 simulation repetitions.

5.1 Estimation with spillover effects

In this subsection we examine the finite sample performance of our estimation procedure proposed in Section 2.2. The model considered here is similar to Li (2017), where $y_{i,t}(0)$ follows a factor model structure. We show both stationary and $\mathcal{I}(1)$ case.

5.1.1 Stationary case

The underlying factor model is

$$y_{i,t}(0) = \eta_t + \lambda_t' \mu_i + \epsilon_{i,t}, \tag{31}$$

Table 1: Treatment effect estimation with stationary common factors.

	$N = 10$			$N = 30$			$N = 50$		
	$T = 15$	50	200	15	50	200	15	50	200
No spillover effects									
SCM	-0.062	0.011	-0.003	0.114	-0.005	0.016	0.037	-0.041	-0.033
	(2.113)	(1.249)	(1.586)	(1.642)	(1.244)	(1.273)	(1.408)	(1.290)	(1.182)
SP	-0.077	0.013	0.018	0.091	-0.012	0.010	0.042	-0.031	-0.040
	(2.618)	(1.417)	(1.710)	(1.974)	(1.362)	(1.486)	(1.741)	(1.516)	(1.270)
Concentrated spillover effects									
SCM	-1.326	-0.986	-1.333	-0.756	-0.880	-1.543	-1.492	-1.070	-0.796
	(2.714)	(1.451)	(2.065)	(1.958)	(1.654)	(1.392)	(1.912)	(1.638)	(1.461)
SP	0.267	0.025	0.140	0.248	0.038	0.025	-0.133	-0.055	0.110
	(2.554)	(1.425)	(1.756)	(1.897)	(1.435)	(1.250)	(1.700)	(1.581)	(1.408)
Spreadout spillover effects									
SCM	-2.378	-1.910	-2.114	-2.245	-1.859	-2.398	-2.147	-2.112	-2.154
	(2.493)	(1.470)	(1.696)	(2.029)	(1.472)	(1.369)	(1.791)	(1.538)	(1.313)
SP	-0.048	0.007	0.029	0.090	-0.025	0.018	0.037	-0.048	-0.028
	(2.740)	(1.438)	(2.061)	(2.231)	(1.296)	(1.602)	(1.643)	(1.450)	(1.290)

Notes: The numbers without parentheses are empirical bias in simulation. The ones with parentheses are empirical variance. SCM is the standard synthetic control method assuming no spillover effects. SP is the estimation procedure proposed in this paper that takes spillover effects into account. *No spillover effects* stands for the cases where the true DGP has no spillover effects. *Concentrated spillover effects* is the case where $1/3$ of the control units receive a spillover effect. *Spreadout spillover effects* is the case where $2/3$ of the control units receive a spillover effect of the same level.

where $\lambda_t = (\lambda_{1,t}, \lambda_{2,t}, \lambda_{3,t})'$,

$$\eta_t = 1 + 0.5\delta_{-1} + \nu_{0,t}, \tag{32}$$

$$\lambda_{1,t} = 0.5\lambda_{1,t-1} + \nu_{1,t}, \tag{33}$$

$$\lambda_{2,t} = 1 + \nu_{2,t} + 0.5\nu_{2,t-1}, \tag{34}$$

$$\lambda_{3,t} = 0.5\lambda_{3,t-1} + \nu_{3,t} + 0.5\nu_{3,t-1}, \tag{35}$$

and $\epsilon_{i,t}$ and $\nu_{j,s}$ is i.i.d. $N(0,1)$ for each (i,j,s,t). Each entry of μ_i is drawn from an independent uniform distribution on $[0,1]$ and fixed for each repetition. At $t = T + 1$, the observed outcome is $y_{i,T+1} = y_{i,T+1}(0) + \alpha_i$, where α_i is either treatment effect or spillover effect and is specified below. The treatment effect is set to 5 and the spillover effect is 3.

The empirical bias and variance (in parentheses) of the treatment effect estimator using two methods
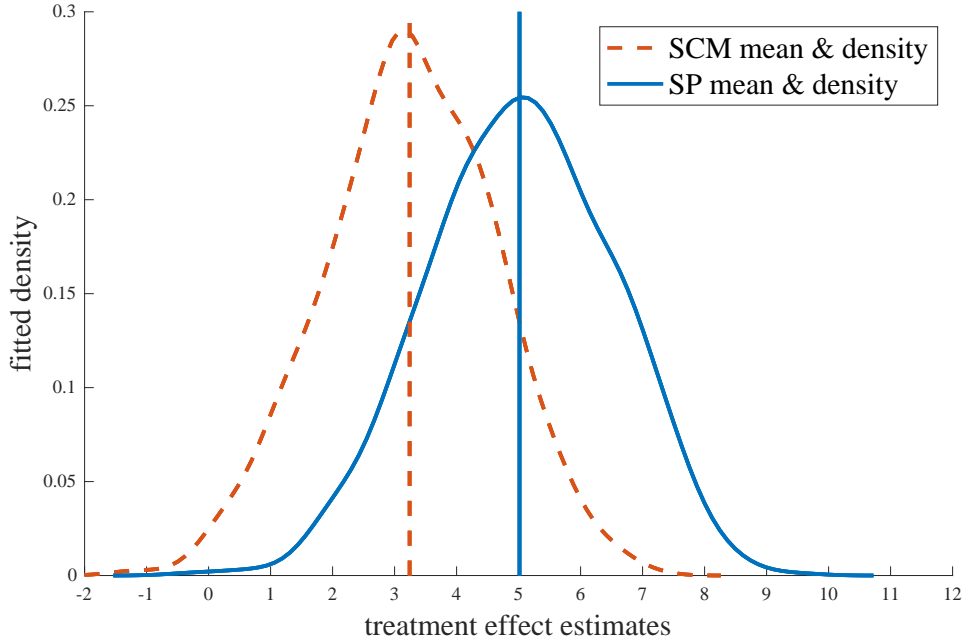
Figure 3: Distribution of treatment effect estimates. The true treatment effect is 5. SCM is using the standard synthetic control method assuming no spillover effects. SP is the estimation procedure proposed in this paper that takes spillover effects into account. Estimates are fitted using kernel density.

are shown in Table 1. We consider three spillover patterns. *No spillover effects* is the case where unit 1 receives a treatment effect of 5 at $t = T + 1$ and other units are not affected. *Concentrated spillover effects* is the case where 1/3 of the control units receive a spillover effect of 3. *Spreadout spillover effects* is the case where 2/3 of the control units receive a spillover effect of 3. SCM is the original synthetic control method, and SP is the corrected synthetic control method proposed in Section 2.3. Throughout the simulations we assume the coverage of spillover effect is known, but not other information, so A is constructed as in Example 3. For *No spillover effects*, we are being conservative in our use of the SP estimator and run it as if 1/3 of the control units are exposed to spillover effects.

To better compare results, we fit the simulation results using kernel density for the $(N, T) = (10, 50)$ case with concentrated spillover effects and plot it in Figure 3.

5.1.2 $\mathcal{I}(1)$ case

For the $\mathcal{I}(1)$ case, the underlying factor model follows

$$y_{i,t}(0) = \lambda_t' \mu_i + \epsilon_{i,t}, \tag{36}$$

18

Table 2: Treatment effect estimation with $\mathcal{I}(1)$ common factors.

	$N = 10$			$N = 30$			$N = 50$		
	$T = 15$	50	200	15	50	200	15	50	200
No spillover effects									
SCM	-0.023	-0.018	-0.043	0.036	-0.088	-0.031	0.041	0.038	-0.038
	(1.873)	(1.642)	(1.772)	(1.708)	(1.539)	(1.900)	(1.915)	(1.810)	(1.866)
SP	-0.021	-0.057	-0.017	0.037	-0.053	-0.044	0.007	0.013	-0.017
	(2.460)	(2.249)	(4.523)	(2.116)	(2.121)	(2.184)	(2.308)	(1.849)	(1.952)
Concentrated spillover effects									
SCM	-1.185	-1.400	-2.234	-1.206	-2.026	-1.954	-1.316	-1.408	-2.325
	(2.421)	(1.854)	(1.856)	(2.269)	(1.921)	(2.079)	(2.449)	(2.043)	(1.976)
SP	-0.021	-0.057	-0.017	0.037	-0.053	-0.044	0.007	0.013	-0.017
	(2.460)	(2.249)	(4.523)	(2.116)	(2.121)	(2.184)	(2.308)	(1.849)	(1.952)
Spreadout spillover effects									
SCM	-2.088	-2.599	-2.885	-2.233	-2.536	-2.465	-2.219	-2.402	-2.889
	(2.390)	(1.779)	(1.795)	(2.101)	(1.759)	(2.037)	(2.249)	(1.921)	(1.900)
SP	-0.029	0.027	-0.022	0.047	-0.008	0.010	0.022	0.006	-0.045
	(2.452)	(3.447)	(7.367)	(2.357)	(2.412)	(2.740)	(2.418)	(2.279)	(2.712)

Notes: The numbers without parentheses are empirical bias in simulation. The ones with parentheses are empirical variance. SCM is the standard synthetic control method assuming no spillover effects. SP is the estimation procedure proposed in this paper that takes spillover effects into account. *No spillover effects* stands for the cases where the true DGP has no spillover effects. *Concentrated spillover effects* is the case where 1/3 of the control units receive a spillover effect. *Spreadout spillover effects* is the case where 2/3 of the control units receive a spillover effect of the same level.

where $\lambda_t = (\lambda_{1,t}, \lambda_{2,t}, \lambda_{3,t})'$,

$$\lambda_{1,t} = \lambda_{1,t-1} + 0.5\nu_{1,t}, \tag{37}$$

$$\lambda_{2,t} = \lambda_{2,t-1} + 0.5\nu_{2,t}, \tag{38}$$

$$\lambda_{3,t} = 0.5\lambda_{3,t-1} + \nu_{3,t}, \tag{39}$$

and $\epsilon_{i,t}$ and $\nu_{j,s}$ follows i.i.d. $N(0,1)$ for each (i,j,s,t). The factor loadings are constructed such that condition CO is satisfied. Namely, we let $\mu_1 = (1,0,0)'$, $\mu_2 = (0,1,0)'$, $\mu_3 = (1,0,0)'$, $\mu_4 = (0,1,0)'$, and for μ_j with $j = 5, \ldots, N$, we draw independent uniform distribution on $[0,1]$ for each entry and then normalize each loading vector such that three entries of each μ_j sum up to one. The constructed factor loadings are fixed for each repetition while other settings are same as the stationary case. The results are shown in Table 2.

Table 3: Empirical rejection rate of testing for treatment effects under null.

| | $N = 10$ | | | $N = 30$ | | | $N = 50$ | | |
	$T = 15$	50	200	15	50	200	15	50	200
No spillover effects									
Placebo	0.000	0.000	0.000	0.072	0.053	0.062	0.034	0.031	0.040
Andrews	0.076	0.061	0.060	0.108	0.082	0.065	0.141	0.078	0.072
SP	0.048	0.049	0.058	0.055	0.064	0.052	0.066	0.046	0.059
Concentrated spillover effects									
Placebo	0.000	0.000	0.000	0.066	0.046	0.116	0.035	0.029	0.026
Andrews	0.411	0.207	0.224	0.417	0.279	0.346	0.519	0.346	0.184
SP	0.065	0.050	0.043	0.111	0.069	0.061	0.109	0.092	0.054
Spreadout spillover effects									
Placebo	0.000	0.000	0.000	0.129	0.063	0.147	0.060	0.059	0.072
Andrews	0.576	0.478	0.399	0.685	0.563	0.616	0.741	0.621	0.544
SP	0.036	0.035	0.042	0.034	0.042	0.046	0.030	0.042	0.044

Notes: SP is the estimation procedure proposed in this paper that takes spillover effects into account. *No spillover effects* stands for the cases where the true DGP has no spillover effects. *Concentrated spillover effects* is the case where 1/3 of the control units receive a spillover effect. *Spreadout spillover effects* is the case where 2/3 of the control units receive a spillover effect of the same level.

5.2 Test for treatment effects

In this section we compare test procedures against the null hypothesis $H_0 : \alpha_1 = 0$, i.e. the treatment effect is zero. The results are shown in Table 3 and Table 4. The DGP is exactly the same as in Section 5.1.1 (the stationary case), except that $\alpha_1 = 0$ (the null) for Table 3 and $\alpha_1 = 5$ (the alternative) for Table 4. Placebo test is as in Abadie and Gardeazabal (2003) and Hahn and Shi (2017). Andrews' test is as in Andrews (2003). SP is the spillover-adjust test proposed in Section 3.2.

Among the three testing procedures, SP test has correct sizes and outperforms the other two methods in power. Placebo test has correct sizes in some cases but has lower power, and Andrews' test over-rejects under null. The reasons are discussed in Section 3.3.

5.3 Test for existence of spillover effects

In this section we examine the power of the proposed test against the null hypothesis that there are no spillover effects. We also look into its behavior when the range of the spillover effect is not correctly specified. In this set of experiments, the level of spillover effects varies from 0 to 2, corresponding to the strength of alternative hypotheses. We set $(N, T) = (20, 50)$ and $\alpha_1 = 5$. There are 9 units that are affected by spillover effects. Other settings follow exactly as in Section 5.1.1 (the stationary case). The

Table 4: Empirical rejection rate of testing for treatment effects under alternative.

	$N = 10$			$N = 30$			$N = 50$		
	$T = 15$	50	200	15	50	200	15	50	200
No spillover effects									
Placebo	0.000	0.000	0.000	0.908	0.939	0.966	0.922	0.936	0.931
Andrews	0.797	0.948	0.926	0.785	0.901	0.983	0.797	0.972	0.827
SP	0.835	0.956	0.923	0.823	0.937	0.965	0.839	0.964	0.993
Concentrated spillover effects									
Placebo	0.000	0.000	0.000	0.461	0.502	0.448	0.465	0.434	0.464
Andrews	0.651	0.765	0.329	0.704	0.754	0.542	0.680	0.746	0.737
SP	0.860	0.932	0.991	0.957	0.918	0.967	0.834	0.816	0.853
Spreadout spillover effects									
Placebo	0.000	0.000	0.000	0.348	0.378	0.331	0.305	0.255	0.294
Andrews	0.337	0.403	0.277	0.563	0.414	0.278	0.406	0.309	0.343
SP	0.866	0.978	0.981	0.969	0.950	0.991	0.909	0.985	0.974

Notes: SP is the estimation procedure proposed in this paper that takes spillover effects into account. *No spillover effects* stands for the cases where the true DGP has no spillover effects. *Concentrated spillover effects* is the case where 1/3 of the control units receive a spillover effect. *Spreadout spillover effects* is the case where 2/3 of the control units receive a spillover effect of the same level.

model for the range of spillover is as in Example 3.

The empirical rejection rates against various levels of spillover effects using our method proposed in Section 3.2 are plotted in Figure 4. Here *Include too few* misses half of the units that are actually affected by the treatment (assuming that unit 1 as well as four other units are affected), *Correct specification* assumes we know exactly which units are affected, and *Include too many* assumes 15 units are affected in estimation, 5 of which are actually not affected by spillover effects.

The simulation results show that the proposed test is quite robust to model misspecification. Among the three cases, *Include too many* is still a correct specification but is supposed to be more conservative, so it has less power than *Correct specification* does. The range of spillover effects is misspecified in *Include too few*, but the test is still correctly sized under the null[4] and has reasonable power under alternatives.

6 Empirical Example

To demonstrate our method, we use it on the classic SCM example from Abadie et al. (2010) (ADH), which looks at the effect of Proposition 99 on California cigarette consumption. In this section, we will

---

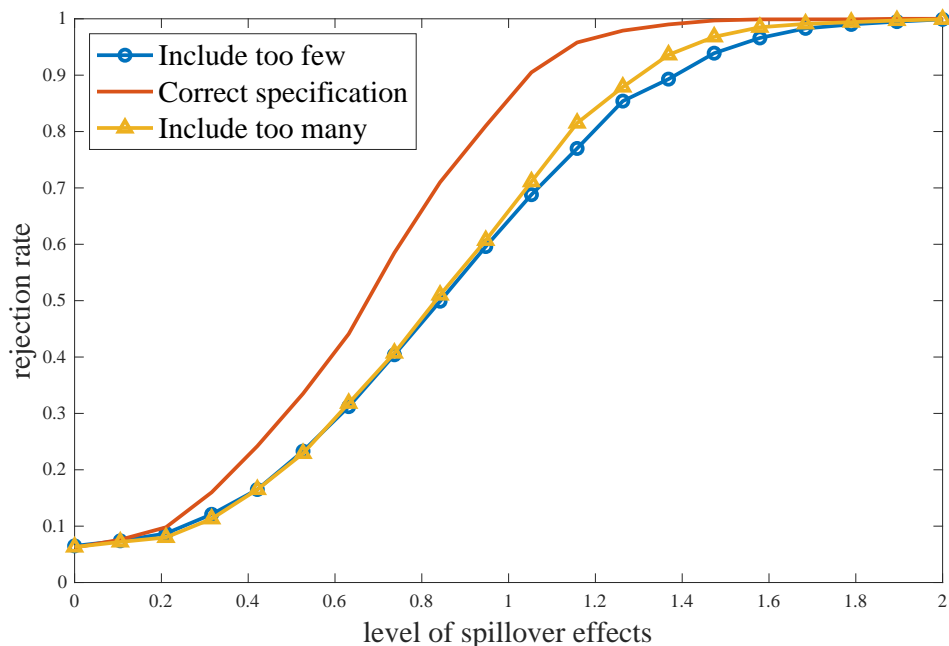[4]The model is always correctly specified under null.

Figure 4: Empirical rejection rate of testing for existence of spillover effects. There are 20 units in total and half of them are affected by the treatment. *Include too few* is assuming only 5 of them are affected by the treatment. *Correct specification* assumes the researcher knows exactly which set of units are affected. *Include too many* assumes 15 units are affected, 5 of which are in fact not affected.

walk through the results from our method, with interruptions to point out key features and issues.

Proposition 99 intended to disincentivize smoking, which was primarily achieved by introducing a $0.25 tax on each pack of cigarettes. By measuring sales in California, ADH and others have attempted to determine the effect of the policy on smoking rates. However, traditional SCM is not guaranteed to produce an unbiased treatment effect estimator in the presence of spillover effects. In this tobacco control program example, we are concerned about two kinds of spillover effects. The first spillover is based on concerns about "leakage". A common problem with cigarette taxes is that measured local consumption might fall as people move their purchasing behavior across legal boundaries. In order to accommodate this, we allowed for a spillover affecting states neighboring California and a spillover affecting states which a state away from California. The second spillover type we considered was a cultural change. If tobacco is discouraged in California, it might reduce the cultural appeal of smoking. Reasoning that the northeast is culturally close to the west coast, we allowed for the northeastern states to experience this cultural spillover.

One might also think that there could be a policy contamination whereby culturally close states also enact policies with similar targets. Our method can allow for this kind of spillover in our estimation. However, the initial paper took that type of problem into account, and 12 states which experienced
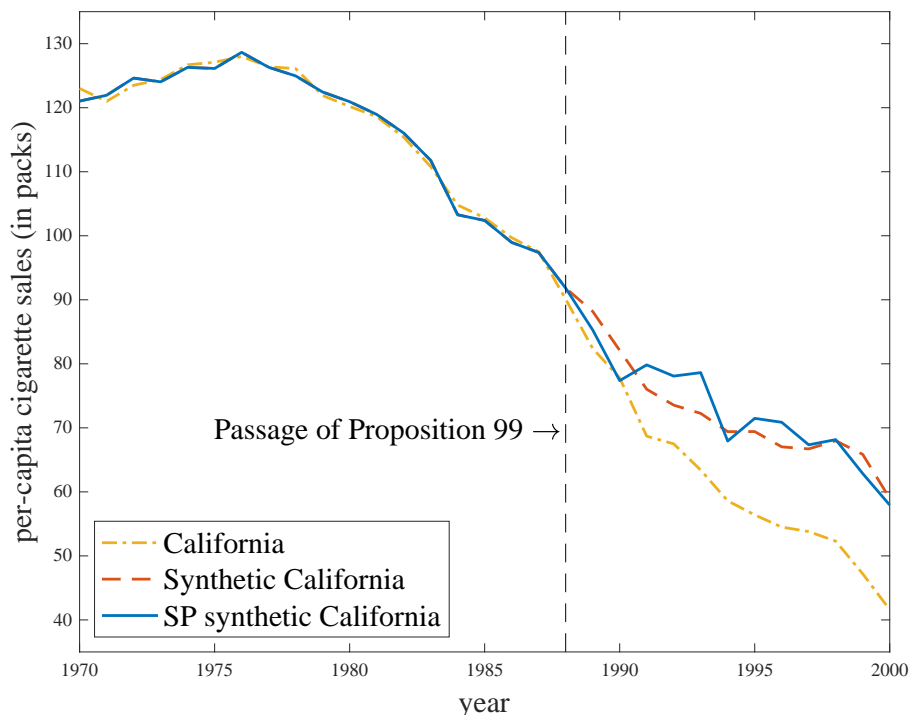
Figure 5: Trends in per-capita cigarette sales: California, synthetic California, and spillover-adjusted synthetic California. SP synthetic California is using our estimation procedure, which accounts for spillover effects. The vertical line indicates the start of treatment.

legislative changes in the ensuing years were removed in that paper (and thus in our data).

The data used is per capita cigarette consumption in 38 of the 50 states running from 1970 to 2000. Twelve states were removed from the data because of concerns that they were either contaminated, or received treatment later on. In 1989 California enacted Proposition 99, so all periods from 1989 onwards are considered post-treatment periods. We replicate this program evaluation using the method introduced in previous sections, allowing for possible spillover effects. We use the spillover structure as in Example 3. That is, we allow for arbitrary spillover effects in those geographically close and culturally similar states as described in the last paragraph, but not the others. We also perform hypothesis testing on both treatment effects and spillover effects.

The results are shown in Figure 5 and Figure 6. The method in Abadie et al. (2010) is indexed by SCM and our method is SP. Figure 5 shows the "synthetic California" and Figure 6 elaborates on this by specifically looking at the estimated treatment effects. The error bars are built using the methods described in this paper, at a significance level of 90%. We do not use a 95% significance level because there are only 19 pre-treatment periods.

As Figure 5 shows, our estimated consumption in the "synthetic California" does not differ qualitatively from what a standard SCM would predict. Quantitatively, Figure 6 shows that our results are more consistent with an addiction story, that tobacco consumption is addictive and should not fall immediately
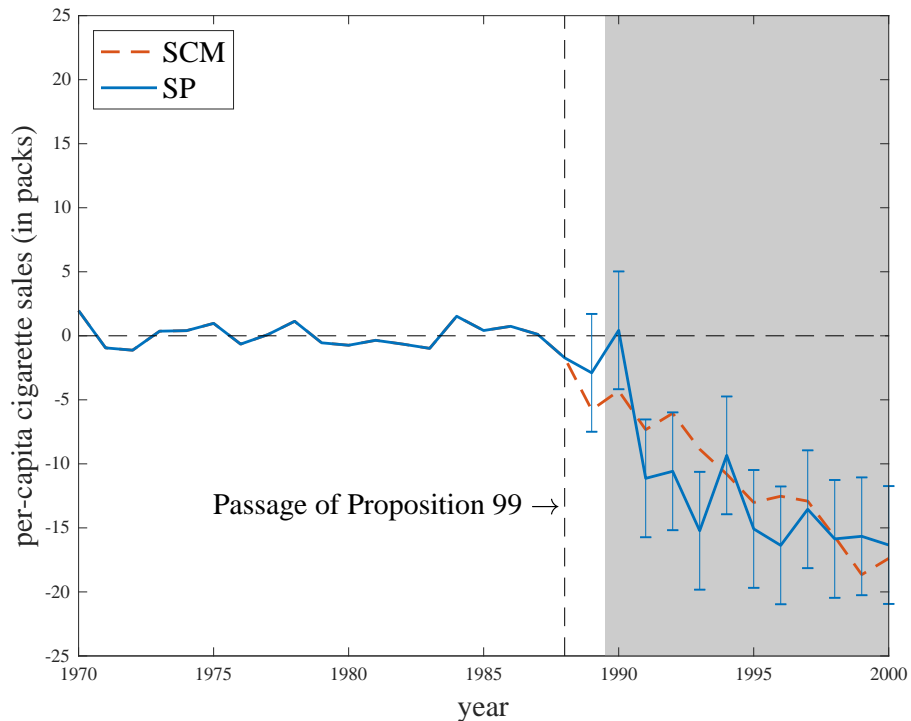
Figure 6: Per-capita cigarette sales gap between California and (spillover-adjusted) synthetic California (with 90% confidence interval). The lines to the right of passage of Proposition 99 are treatment effect estimates. SCM is obtained by using standard synthetic control method. SP is using our estimation procedure, which accounts for spillover effects. Shaded area denotes our test rejects there is no spillover effects in those years.

after the policy. From the tests of spillover effects (shaded area of Figure 6), we see that likely there were substantial spillover effects, which in some periods lead to statistically significant changes in the treatment effect estimates. For example, the SCM estimate of year 1990 lies outside our confidence interval, which potentially results from the over-estimation of scale of the treatment effects in the presence of spillover effects.

7   Conclusion

The synthetic control method is a powerful tool in treatment effect estimation in the panel data settings, but it does not work in the presence of spillover effects. In this paper, we relax this assumption and propose an estimation and testing procedure that is robust to the presence of spillover effects. Our method requires specification of the spillover structure, which can be relatively weak (Example 3). We consider both stationary and cointegrated cases and show that our estimators are asymptotically unbiased. We develop a testing procedure based on Andrews (2003)'s end-of-sample instability tests and show that it is asymptotically unbiased. Our methods can be extended to cases with multiple treated units and multiple post-treatment periods. Simulation results certify the validity of our estimation and testing procedure

in the presence of spillover effects. The simulations also indicate that our testing procedure is relatively robust to misspecification of the spillover structure. Finally, we illustrate our method by applying it to Abadie et al. (2010)'s California tobacco control program data.

Appendix

Proof of Lemma 1. First assume Condition ID and Condition ST holds. The proof follows Ferman and Pinto (2016), except that we do not assume that there is a set of weights that reconstruct the factor loadings and belong to the simplex.

We first show part (b). It suffices to show $|\hat{a}_i - a_i| = o_p(1)$ and $\|\hat{b}_i - b_i\| = o_p(1)$ for each i, i.e. a_i and b_i are well-defined. We show it for the $i = 1$ case and other cases follow the same strategy. Let $\bar{y}_j = T^{-1} \sum_{t=1}^{T} y_{j,t}$. Write down an (equivalent) optimization problem

$$\hat{v} = \arg\min_{v \in V} \left((y_{1,t} - \bar{y}_1) - \sum_{j=2}^{N} (y_{j,t} - \bar{y}_j) v_j \right)^2, \tag{40}$$

where $V = \{v = (v_2, \ldots, v_N) \in \mathbb{R}_+^{N-1} : \sum_{j=2}^{N} v_j = 1\}$. The objective is strictly convex (with probability approaching one), so the solution is unique. Note that it implies $\hat{b}_1$ is numerically equivalent to $(0, \hat{v}')'$, otherwise the minimization problem in forming $\hat{a}_1$ and $\hat{b}_1$ may have a lower objective evaluated at $(\bar{y}_1 - \sum_{j=2}^{N} \bar{y}_j \hat{v}_j, 0, \hat{v}')'$. Now we let $\hat{Q}(v)$ denote the objective function such that

$$\hat{Q}(v) = \frac{1}{T} \sum_{t=1}^{T} \left((y_{1,t} - \bar{y}_1) - \sum_{j=2}^{N} (y_{j,t} - \bar{y}_j) v_j \right)^2, \tag{41}$$

and its population analog be

$$Q(v) = \begin{bmatrix} -1 \\ v \end{bmatrix}' \Omega_y \begin{bmatrix} -1 \\ v \end{bmatrix}. \tag{42}$$

Let v_0 be a minimizer of $Q(v)$ in V. We verify the conditions for consistency (see Newey and McFadden (1994), Theorem 2.1): (i) Since Ω_y is positive definite, $Q(v)$ is strictly convex. Also, V is convex. Therefore, $Q(v)$ is uniquely minimized at v_0. (ii) V is compact, since it is a $(N-1)$-dimensional simplex.

(iii) $Q(v)$ is continuous, since it has a quadratic form. (iv) To see uniform convergence, note

$$
\sup_{v \in V} |\hat{Q}(v) - Q(v)| = \sup_{v \in V} \left\| \begin{bmatrix} -1 \\ v \end{bmatrix}' \left(\frac{1}{T} \sum_{t=1}^{T} (Y_t - \bar{Y})(Y_t - \bar{Y})' - \Omega_y \right) \begin{bmatrix} -1 \\ v \end{bmatrix} \right\|
$$

$$
\leq \sup_{v \in V} \left\| \begin{bmatrix} -1 \\ v \end{bmatrix} \right\|^2 \left\| \frac{1}{T} \sum_{t=1}^{T} (Y_t - \bar{Y})(Y_t - \bar{Y})' - \Omega_y \right\|_F
$$

$$
\leq N \cdot o_p(1)
$$

$$
= o_p(1), \tag{43}
$$

where $\| \cdot \|_F$ is the Frobenius norm. The second inequality is by ergodicity for the second moments. Therefore, $\hat{v} \to_p v_0$. This implies $\|\hat{b}_1 - b_1\| = o_p(1)$. By ergodicity,

$$
\hat{a}_1 = \bar{y}_1 - [\bar{y}_2 \ \bar{y}_3 \ \ldots \ \bar{y}_N] \hat{v} \to_p E[y_{1,t}(0) - Y_t(0)'b_1] = a_1. \tag{44}
$$

This shows part (b) and $E[u_{1,t}] = 0$ by definition of $u_{i,t}$. We also have that $\{u_t\}_{t \geq 1}$ is stationary since it is a linear combination of stationary and ergodic processes. This shows part (a) in Assumption 1.

Part (c) follows from part (b) and the stationarity of $\{Y_{T+1}(0)\}_{T \geq 1}$. Part (d) follows by Condition ID. Thus, Assumption 1 holds under Condition ID and Condition ST.

Now we instead assume Condition ID and Condition CO holds.

We first show part (c). We will show $\|Y_{T+1}(0)'(\hat{b}_1 - b_1)\| = o_p(1)$ and other i's follows the same strategy. We follows Li (2017)'s strategy of treating the synthetic control weight estimator as a projection of the OLS estimator onto a closed convex set. Namely, for some positive definite matrix $D \in \mathbb{R}^N$, let $\mathbb{R}^N$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_D$ such that for $\theta_1, \theta_2 \in \mathbb{R}^N$,

$$
\langle \theta_1, \theta_2 \rangle_D = \theta_1' D \theta_2. \tag{45}
$$

The norm $\| \cdot \|_D$ is defined accordingly, i.e. $\|\theta\|_D = \sqrt{\theta' D \theta}$, for $\theta \in \mathbb{R}^N$. For a closed convex set $\Lambda \subset \mathbb{R}^N$, define a projection Π_D such that for each $\theta \in \mathbb{R}^N$, $\Pi_D \theta = \arg\min_{\theta' \in \Lambda} \|\theta - \theta'\|_D$. Zarantonello (1971) shows that for each $\theta, \theta' \in \mathbb{R}^N$,

$$
\|\Pi_D \theta - \Pi_D \theta'\|_D \leq \|\theta - \theta'\|_D. \tag{46}
$$

With some abuse of notation, let $x_t = Y_t - T^{-1} \sum_{s=1}^{T} Y_s$. Then, $\hat{b}_1$ is the synthetic control weight estimators of regressing $(y_{1,t} - T^{-1} \sum_{s=1}^{T} y_{1,s})$ on x_t, subject to $\{0\} \times \Delta_{N-1}$ with Δ_{N-1} being an $(N-1)$-dimensional simplex. Let $\tilde{b}_1$ be the OLS estimator of regressing $(y_{1,t} - T^{-1} \sum_{s=1}^{T} y_{1,s})$ on x_t. Let $\Sigma_T = T^{-1} \sum_{t=1}^{T} x_t x_t'$.

Appendix A.2 in Li (2017) establishes that $\hat{b}_1 = \Pi_{\Sigma_T} \tilde{b}_1$. Thus, we have

$$
\begin{aligned}
\|\hat{b}_1 - b_1\| &= \|\Sigma_T^{-1/2} \Sigma_T^{1/2} (\hat{b}_1 - b_1)\| \\
&\leq \|\Sigma_T^{-1/2}\|_F \cdot \|\Sigma_T^{1/2} (\hat{b}_1 - b_1)\| \\
&= \|\Sigma_T^{-1/2}\|_F \cdot \|\hat{b}_1 - b_1\|_{\Sigma_T} \\
&= \|\Sigma_T^{-1/2}\|_F \cdot \|\Pi_{\Sigma_T} \tilde{b}_1 - \Pi_{\Sigma_T} b_1\|_{\Sigma_T} \\
&\leq \|\Sigma_T^{-1/2}\|_F \cdot \|\tilde{b}_1 - b_1\|_{\Sigma_T} \\
&= \|\Sigma_T^{-1/2}\|_F \cdot \|\Sigma_T^{1/2}\|_F \cdot \|\tilde{b}_1 - b_1\| \\
&= O_p(1) o_p(T^{-1/2}) \\
&= o_p(T^{-1/2}),
\end{aligned}
\tag{47}
$$

where $\|\cdot\|_F$ is the Frobenius norm of a matrix. The third equality is because $b_1 \in \{0\} \times \Delta_{N-1}$. The second inequality is by (46). To see the fifth equality, note

$$
\Sigma_T = T \left(\frac{1}{T^2} \sum_{t=1}^T Y_t Y_t' - \left(\frac{1}{T^{3/2}} \sum_{t=1} Y_t \right) \left(\frac{1}{T^{3/2}} \sum_{t=1} Y_t \right)' \right),
\tag{48}
$$

so

$$
\|\Sigma_T^{-1/2}\|_F \cdot \|\Sigma_T^{1/2}\|_F = \operatorname{tr}(\Sigma_T^{-1}) \operatorname{tr}(\Sigma_T) = O_p(1) \cdot \frac{1}{T} \cdot T \cdot O_p(1) = O_p(1),
\tag{49}
$$

where the second equality is standard results for $\mathcal{I}_1$ process (see part (g) and (i) of Proposition 18.1 in Hamilton (1994) for example). Also, $\|\tilde{b}_1 - b_1\| = o_p(T^{-1/2})$ is by Proposition 19.2 in Hamilton (1994). This shows (47). Apply part (a) of Proposition 18.1 in Hamilton (1994), we have

$$
\|Y_{T+1}(0)'(\hat{b}_1 - b)\| = \|(T^{-1/2} Y_{T+1}(0))'(T^{-1/2}(\hat{b}_1 - b))\| = O_p(1) o_p(1) = o_p(1).
\tag{50}
$$

Now we show part (b). Again, it suffices to show $|\hat{a}_i - a_i| = o_p(1)$ and $\|\hat{b}_i - b_i\| = o_p(1)$. We consider the $i = 1$ case and other cases follow the same strategy. We have showed $\|\hat{b}_i - b_i\| = o_p(1)$ in part (c) of the proof. Section A.6.1 in Ferman and Pinto (2016) establishes that

$$
[\mu_1^1 \ \mu_2^1 \ \ldots \ \mu_N^1](b_1 - e_1) = 0,
\tag{51}
$$

where e_i is the unit vector with one at the i-th entry. Thus,

$$\hat{a}_1 = [\bar{y}_1 \ \bar{y}_2 \ \dots \ \bar{y}_N](e_1 - \hat{b}_1)$$

$$= [\bar{y}_1 \ \bar{y}_2 \ \dots \ \bar{y}_N](e_1 - b_1) + [\bar{y}_1 \ \bar{y}_2 \ \dots \ \bar{y}_N](b_1 - \hat{b}_1)$$

$$= \left\{ \frac{1}{T} \sum_{t=1}^{T} \left((\lambda_t^0)'[\mu_1^0 \ \dots \ \mu_N^0] + [\epsilon_{1,t} \ \dots \ \epsilon_{N,t}] \right) \right\} (e_1 - b_1) + \left(\frac{1}{\sqrt{T}}[\bar{y}_1 \ \bar{y}_2 \ \dots \ \bar{y}_N] \right) \sqrt{T}(b_1 - \hat{b}_1)$$

$$= E[\lambda_t^0]'[\mu_1^0 \ \dots \ \mu_N^0](e_1 - b_1) + o_p(1) + O_p(1)o_p(1)$$

$$\to_p E[\lambda_t^0]'[\mu_1^0 \ \dots \ \mu_N^0](e_1 - b_1).$$

$$= a_1 \qquad\qquad (52)$$

The third equality is by (51). The fourth equality is by stationarity of $\{(\lambda_t^0, \epsilon_t)\}_{t \geq 1}$ and results in part (d) of the proof. This shows part (b) of the Assumption 1 .

Combining (51) and (52), we have part (a) in Assumption 1. Part (d) is assumed by Condition ID. □

Proof of Theorem 1. Using formula of $\hat{\gamma}$ in Equation (14), we have

$$\hat{\gamma} = (A'\hat{M}A)^{-1}A'(I - \hat{B})'((I - \hat{B})Y_{T+1}(0) + (I - \hat{B})\alpha - \hat{a})$$

$$= (A'\hat{M}A)^{-1}A'(I - \hat{B})'(u_{T+1} + (B - \hat{B})Y_{T+1}(0) + (a - \hat{a}) + (I - \hat{B})A\gamma)$$

$$= (A'\hat{M}A)^{-1}A'(I - \hat{B})'u_{T+1} + o_p(1) + o_p(1) + \gamma. \qquad\qquad (53)$$

The first equality is by $Y_{T+1} = Y_{T+1}(0) + \alpha$. The second equation is because $Y_{T+1}(0) = a + BY_{T+1}(0) + u_{T+1}$. The third equation is by (b) and (c) in Assumption 1. Therefore,

$$\hat{\alpha} - (\alpha + Gu_{T+1}) = A(A'\hat{M}A)^{-1}A'(I - \hat{B})'u_{T+1} + A\gamma + o_p(1) - \alpha - Gu_{T+1}$$

$$= (A(A'\hat{M}A)^{-1}A'(I - \hat{B}) - G)'u_{T+1} + o_p(1)$$

$$= o_p(1)O_p(1) + o_p(1)$$

$$= o_p(1). \qquad\qquad (54)$$

The third equality is by (b) in Assumption 1 and stationarity of $\{u_t\}_{t \geq 1}$. □

Proof of Proposition 1. The proof for the first half of the proposition is similar to the proof for Theorem 1, and thus is omitted. Too see the second half, note

$$Cov[G_W u_{T+1}] = A(Q'WQ)^{-1}Q'W\Omega WQ(Q'WQ)^{-1}A' \qquad\qquad (55)$$

and

$$Cov[G^e u_{T+1}] = A(Q'\Omega Q)^{-1}A', \qquad\qquad (56)$$

28

where $Q = (I - B)A$. It suffices to show $((Q'WQ)^{-1}Q'W\Omega WQ(Q'WQ)^{-1} - (Q'\Omega Q)^{-1})$ is positive semi-definite. Note that the first term is asymptotic variance of using W as the weighting matrix in GMM exercise and the second term is the one using the efficient weighting matrix (see Proposition 3.5 in Hayashi (2000)). Thus, $(Cov[G_W u_{T+1}] - Cov[G^e u_{T+1}])$ is positive semi-definite. $\square$

Proof of Lemma 2. Since Assumption 3 implies Assumption 2, we only need to show Lemma 3. $\square$

Proof of Theorem 2. We follow the proof of Theorem 2 in Andrews and Kim (2006). Let

$$
\begin{aligned}
L_{1,T}(\epsilon) &= \left\{ \|C_T(\hat{\beta}_1 - \beta_1)\| \leq \epsilon, \max_{t=1,\ldots,T} \|C_T(\hat{\beta}_1^{(t)} - \beta_1)\| \leq \epsilon \right\}, \\
L_{2,T}(c) &= \left\{ \max_{t \leq T+1} \|C_T^{-1}x_t\| \leq c \right\}.
\end{aligned}
\tag{57}
$$

By Assumption 2(d), there exists a positive sequence $\{\epsilon_T\}_{T \geq 1}$ such that $\epsilon_T \to 0$ and $\Pr(L_{1,T}(\epsilon_T)) \to 1$. Let $c_T = 1/\sqrt{\epsilon_T}$. So we have $c_T \to \infty$ and $c_T \epsilon_T \to 0$. By Assumption 2(c), we must have $\Pr(L_{2,T}(c_T)) \to 1$. Let $L_T = L_{1,T}(\epsilon_T) \cap L_{2,T}(c_T)$, then we have $\Pr(L_T) \to 1$ and $\Pr(L_T^c) \to 0$.

Suppose L_T holds. Then, for $\beta = \hat{\beta}_1$ or $\beta = \hat{\beta}_1^{(t)}$ for some $t = 1, \ldots, T$, we have

$$
\begin{aligned}
|P_t(\beta) - P_t(\beta_1)| &= \left| (\beta - \beta_1)' x_t x_t' (\beta - \beta_1) - 2x_t'(\beta - \beta_1)u_{1,t} \right| \\
&= \left| (\beta - \beta_1)' C_T'(C_T')^{-1} x_t x_t' C_T^{-1} C_T(\beta - \beta_1) - 2x_t' C_T^{-1} C_T(\beta - \beta_1)u_{1,t} \right| \\
&\leq \|C_T(\beta - \beta_1)\|^2 \|C_T^{-1}x_t\|^2 + 2\|C_T^{-1}x_t\| \|C_T(\beta - \beta_1)\| |u_{1,t}| \\
&\leq \epsilon_T^2 c_T^2 + 2\epsilon_T c_T |u_{1,t}|.
\end{aligned}
\tag{58}
$$

Define $g_t(\epsilon_T, c_T) = \epsilon_T^2 c_T^2 + 2\epsilon_T c_T |u_{1,t}|$. Note that $g_t(\epsilon_T, c_T)$ is identically distributed across t for a fixed T, by Assumption 2(a).

We first prove part (a). Let x be some continuous point of distribution function of $P_{T+1}(\beta_1)$. Then,

$$
\begin{aligned}
\Pr(P_{T+1}(\hat{\beta}_1) \leq x) &= \Pr(\{P_{T+1}(\hat{\beta}_1) \leq x\} \cap L_T) + \Pr(\{P_{T+1}(\hat{\beta}_1) \leq x\} \cap L_T^c) \\
&\leq \Pr(P_{T+1}(\hat{\beta}_1) \leq x + g_t(\epsilon_T, c_T)) + \Pr(L_T^c) \\
&\leq \Pr(P_{T+1}(\beta_1) \leq x) + o(1).
\end{aligned}
\tag{59}
$$

To see the last equality, pick $\epsilon > 0$. By continuity, $\exists \delta > 0$ such that for each $y \in (x - \delta, x + \delta)$, $|\Pr(P_{T+1}(\beta_1) \leq y) - \Pr(P_{T+1}(\beta_1) \leq x)| < \epsilon$. Therefore,

$$
\begin{aligned}
\Pr(P_{T+1}(\hat{\beta}_1) \leq x + g_t(\epsilon_T, c_T)) &= \Pr(\{P_{T+1}(\hat{\beta}_1) \leq x + g_t(\epsilon_T, c_T)\} \cap \{|g_t(\epsilon_T, c_T)| \geq \delta\}) \\
&\quad + \Pr(\{P_{T+1}(\hat{\beta}_1) \leq x + g_t(\epsilon_T, c_T)\} \cap \{|g_t(\epsilon_T, c_T)| < \delta\}) \\
&\leq \Pr(|g_t(\epsilon_T, c_T)| \geq \delta) + \Pr(P_{T+1}(\hat{\beta}_1) \leq y) \\
&< \Pr(P_{T+1}(\beta_1) \leq x) + o(1).
\end{aligned}
\tag{60}
$$

Similarly,

$$\Pr(P_{T+1}(\hat\beta_1) \le x) \ge \Pr(P_{T+1}(\beta_1) \le x) + o(1). \tag{61}$$

This shows part (a).

To see part (b), let $k : \mathbb{R} \to \mathbb{R}$ be a monotonically decreasing and everywhere differentiable function that has bounded derivative and satisfies $k(x) = 1$ for $x \le 0$, $k(x) \in [0,1]$ for $x \in (0,1)$, and $k(x) = 0$ for $x \ge 1$. For example, let $k(x) = \cos(\pi x)/2 + 1/2$ for $x \in (0,1)$. Given some $\{\beta^{(t)}\}_{t=1}^T$, a smoothed df is defined by

$$\hat F_T(x, \{\beta^t\}, h_T) = \frac{1}{T} \sum_{t=1}^T k\left(\frac{P_t(\beta^{(t)}) - x}{h_T}\right), \tag{62}$$

for some sequence of positive constants $\{h_T\}$ such that $h_T \to 0$ and $c_T \epsilon_T / h_T \to 0$. For example, we let $h_T = \epsilon_T^{1/4}$ when $c_T = 1/\sqrt{\epsilon_T}$.

We write

$$|\hat F_{P,T}(x) - F_P(x)| \le \sum_{i=1}^4 D_{i,T}, \tag{63}$$

for

$$D_{1,T} = |\hat F_{P,T}(x) - \hat F_T(x, \{\hat\beta_j\}, h_T)|,$$

$$D_{2,T} = |\hat F_T(x, \{\hat\beta_j\}, h_T) - \hat F_T(x, \{\beta_1\}, h_T)|,$$

$$D_{3,T} = |\hat F_T(x, \{\beta_1\}, h_T) - \hat F_T(x, \{\beta_1\})|, \text{ and}$$

$$D_{4,T} = |\hat F_T(x, \{\beta_1\}, h_T) - F_P(x\})|. \tag{64}$$

We want to show that all four terms vanish. First note that

$$D_{1,T} \le \frac{1}{T} \sum_{t=1}^T \mathbb{1}\left\{\frac{P_t(\hat\beta_1^{(t)}) - x}{h_T} \in (0,1)\right\}. \tag{65}$$

Thus, for any $\delta > 0$,

$$\Pr(D_{1,T} > \delta) \le \Pr(\{D_{1,T} > \delta\} \cap L_T) + \Pr(L_T^c)$$

$$\le \Pr\left(\frac{1}{T} \sum_{t=1}^T \mathbb{1}\left\{P_t(\hat\beta_1^{(t)}) - x \in (-g_t(\epsilon_T, c_T), h_T + g_t(\epsilon_T, c_T)\right\} > \delta\right) + o(1)$$

$$\le \frac{E\mathbb{1}\left\{P_t(\hat\beta_1^{(t)}) - x \in (-g_t(\epsilon_T, c_T), h_T + g_t(\epsilon_T, c_T)\right\}}{\delta} + o(1), \tag{66}$$

where the last inequality is by Markov's inequality. Recall $\Pr(P_1(\beta_1) \ne x) = 1$ and $g_t(\epsilon_T, c_T) \to 0$ a.s., so $\mathbb{1}\{P_t(\beta_1) - x \in \{-g_t(\epsilon_T, c_T), h_T + g_t(\epsilon_T, c_T)\} \to 0$ a.s.. By the dominated convergence theorem, (66) implies $\Pr(D_{1,T} > \delta) \le o(1)$ and thus $D_{1,T} = o_p(1)$.

For $D_{2,T}$, we have

$$
\begin{aligned}
D_{2,T} &= \left| \frac{1}{T} \sum_{t=1}^{T} k' \left(\frac{\tilde{P}_t - x}{h_T} \right) \frac{P_t(\hat{\beta}_1^{(t)}) - P_t(\beta_1)}{h_T} \right| \\
&\leq \frac{\bar{k}}{T} \sum_{t=1}^{T} \frac{g_t(\epsilon_T, c_T)}{h_T}.
\end{aligned}
\tag{67}
$$

The equality is by the mean value theorem and we have $\tilde{P}_t$ lies between $P_t(\hat{\beta}_1^{(t)})$ and $P_t(\beta_1)$. In the inequality, $\bar{k}$ is a bound for the derivative of k. Also, note

$$
E \left[\frac{g_t(\epsilon_T, c_T)}{h_T} \right] = \frac{\epsilon_T^2 c_T^2}{h_T} + 2 \frac{\epsilon_T c_T}{h_T} E|u_{1,t}| = o(1).
\tag{68}
$$

Therefore,

$$
\begin{aligned}
\Pr(D_{2,T} > \delta) &\leq \Pr(\{D_{2,T} > \delta\} \cap L_T) + \Pr(L_T^c) \\
&\leq \Pr \left(\frac{\bar{k}}{T} \sum_{t=1}^{T} \frac{g_t(\epsilon_T, c_T)}{h_T} > \delta \right) + o(1) \\
&\leq \bar{k} \frac{E g_t(\epsilon_T, c_T)}{\delta h_T} \\
&\to 0.
\end{aligned}
\tag{69}
$$

The third inequality is by Markov's inequality. This shows $D_{2,T} = o_p(1)$.

$D_{3,T}$ is similar to the $D_{1,T}$ case. Finally, by stationary and ergodicity of $u_{1,t}$, we have $D_{4,T} = o_p(1)$. This shows part (b).

Now we show part (c). Pick any small ϵ such that $\hat{F}_{P,T}(x) \to_p F_P(x)$ for $x \in (q_{P,1-\tau} - \epsilon, q_{P,1-\tau} + \epsilon)$. Note

$$
\begin{aligned}
\Pr(\hat{q}_{P,1-\tau} > q_{P,1-\tau} + \epsilon) &\leq \Pr(\hat{F}_{P,T}(q_{P,1-\tau} + \epsilon) < 1 - \tau) \\
&= \Pr(\hat{F}_{P,T}(q_{P,1-\tau} + \epsilon) - F_P(q_{P,1-\tau} + \epsilon) < (1 - \tau) - F_P(q_{P,1-\tau} + \epsilon)) \\
&\to 0.
\end{aligned}
\tag{70}
$$

The inequality is by definition of $\hat{q}_{P,1-\tau}$. The convergence is because of part (e) of Assumption 2 and part (b) of Theorem 2. Similarly,

$$
\begin{aligned}
\Pr(\hat{q}_{P,1-\tau} < q_{P,1-\tau} - \epsilon) &\leq \Pr(\hat{F}_{P,T}(q_{P,1-\tau} - \epsilon) \geq 1 - \tau) \\
&= \Pr(\hat{F}_{P,T}(q_{P,1-\tau} - \epsilon) - F_P(q_{P,1-\tau} - \epsilon) \geq (1 - \tau) - F_P(q_{P,1-\tau} - \epsilon)) \\
&\to 0.
\end{aligned}
\tag{71}
$$

Again, the inequality is by definition of $\hat{q}_{P,1-\tau}$, and the convergence is because of part (e) of Assumption

31

2 and part (b) of Theorem 2.

Finally, we show part (d). Under null, P_∞ and $P_1(\beta_1)$ have the same distribution, so $q_{P,1-\tau}$ is $(1-\tau)$-quantile of P_∞. Therefore,

$$
\begin{aligned}
\Pr(P > \hat{q}_{P,1-\tau}) &= 1 - \Pr(P \le \hat{q}_{P,1-\tau}) \\
&= 1 - \Pr(P + (q_{P,1-\tau} - \hat{q}_{P,1-\tau}) \le q_{P,1-\tau}) \\
&\to \alpha,
\end{aligned}
\tag{72}
$$

where the convergence is by combining part (a) and (c). This concludes our proof. □

Proof of Lemma 3. (i) Assume Condition ST holds.

By Lemma 1, part (a) of Assumption 3 holds.

Part (b) is because u_t is a linear combination of $\eta_t, \lambda_t, \epsilon_t$.

For part (c), pick some τ such that $1/(2+\delta) < \tau < 1/2$, where δ is defined in Condition ST. Let

$$
D_T = \begin{bmatrix} 1 & 0 \\ 0 & T^\tau I_N \end{bmatrix}.
\tag{73}
$$

Then, we have

$$
\max_{t \le T+1} \|D_T^{-1} x_t\| = \max_{t \le T+1} \left\| \begin{bmatrix} 1 \\ T^{-\tau} Y_t \end{bmatrix} \right\| = \sqrt{1 + \left(\max_{t \le T+1} \|T^{-\tau} Y_t\| \right)^2}.
\tag{74}
$$

Also, for any $\epsilon > 0$, note

$$
\begin{aligned}
\Pr\left(\max_{t \le T+1} \|T^{-\tau} Y_t\| > \epsilon \right) &= \Pr\left(\bigcup_{t \le T+1} \|Y_t\| > T^\tau \epsilon \right) \\
&\le \left(\sum_{t=1}^{T} \Pr(\|Y_t\| > T^\tau \epsilon) \right) + \Pr(\|Y_{T+1}(0) + \alpha\| > T^\tau \epsilon) \\
&= \frac{T E[\|Y_t\|^{2+\delta}]}{T^{\tau(2+\delta)} \epsilon^{2+\delta}} + o(1) \\
&= o(1).
\end{aligned}
\tag{75}
$$

The second equality is due to Markov inequality and stationarity of $\{Y_{T+1}(0)\}_{t+1}$. The last equality is because $\tau > 1/(2+\delta)$. Combining (74) and (75), we obtain part (c).

For part (d), we use D_T defined in (73). Following the same reasoning as in (47), for each $i = 1, \ldots, N$, we have

$$
\begin{aligned}
\|\hat{b}_i - b_i\| &\le \|\Sigma_T^{-1/2}\|_F \cdot \|\Sigma_T^{1/2}\|_F \cdot \|\tilde{b}_i - b_i\| \\
&= O_p(1) O_p(T^{-1/2}) \\
&= O_p(T^{-1/2}).
\end{aligned}
\tag{76}
$$

32

The first equality is because $\{Y_t(0)\}_{t \geq 1}$ is ergodic for the second moment, and $\tilde{b}_i$ is the OLS estimator for b_i. Thus,

$$
\begin{aligned}
\|D_T(\hat{\beta}_i - \beta_i)\| &= \left\| \begin{bmatrix} 1 & 0 \\ 0 & T^{\tau-1/2}I_N \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & T^{1/2}I_N \end{bmatrix} (\hat{\beta}_i - \beta_i) \right\| \\
&\leq \left\| \begin{bmatrix} 1 & 0 \\ 0 & T^{\tau-1/2}I_N \end{bmatrix} \right\|_F \left\| \begin{bmatrix} \hat{a}_i - a_i \\ \sqrt{T}(\hat{b}_i - b_i) \end{bmatrix} \right\| \\
&= \sqrt{1 + NT^{2\tau-1}} \|O_p(1)\| \\
&= o_p(1).
\end{aligned}
\tag{77}
$$

The second equality is due to (76). The last equality is because $\tau < 1/2$. Therefore,

$$
\|(\hat{\theta} - \theta_0)D_T\|_F = \sqrt{\sum_{i=1}^{N} \|D_T(\hat{\beta}_i - \beta_i)\|^2} = o_p(1).
\tag{78}
$$

Also, since $\hat{\theta}^{(t)} = \hat{\theta}$ for each t,

$$
\max_{t=1,\ldots,T} \|(\hat{\theta}^{(t)} - \theta_0)D_T\|_F = \|(\hat{\theta} - \theta_0)D_T\|_F = o_p(1).
\tag{79}
$$

This shows part (d).

Part (e) is assumed.

Part (f) is trivial if $W_T = I$. Assume now $W_T = (C\hat{G}(T^{-1}\sum_{t=1}^{T} \hat{u}_t \hat{u}_t')\hat{G}'C')^{-1}$. Then,

$$
\frac{1}{T}\sum_{t=1}^{T} \hat{u}_t \hat{u}_t' = (I - \hat{B})\left(\frac{1}{T}\sum_{t=1}^{T} Y_t Y_t'\right)(I - \hat{B})' - (I - \hat{B})\left(\frac{1}{T}\sum_{t=1}^{T} Y_t\right)\hat{a}' - \hat{a}\left(\frac{1}{T}\sum_{t=1}^{T} Y_t'\right)(I - \hat{B})' + \hat{a}\hat{a}'
$$

$$
\to E[u_t u_t'],
\tag{80}
$$

by ergodicity and Assumption 1(b). Therefore, $\hat{W}_T \to_p W = (CGE[u_t u_t']G'C')^{-1}$.

This concludes part (i) of Lemma 3.

(ii) Assume Condition CO holds.

By Lemma 1, Assumption 1 holds. This shows Part (a).

By (51), u_t is a linear combination of λ_t^o and ϵ_t, so $\{u_t\}_{t \geq 1}$ is ergodic and has finite first moment. This shows Part (b).

Now we show Part (c). Let

$$
D_T = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{T} \cdot I_N \end{bmatrix}.
\tag{81}
$$

Then, we have

$$
\begin{aligned}
\max_{t \leq T+1} \|D_T^{-1} x_t\| &= \sqrt{1 + \left(\max_{t \leq T+1} \|T^{-1/2} Y_t\| \right)^2} \\
&\leq \sqrt{1 + \sum_{i=1}^{N} \left(\max_{t \leq T+1} |T^{-1/2} y_{i,t}| \right)^2} \\
&\leq \sqrt{1 + \sum_{i=1}^{N} \left(T^{-1} |\alpha_i| + \max_{t \leq T+1} |T^{-1/2} y_{i,t}(0)| \right)^2} \\
&= \sqrt{1 + \sum_{i=1}^{N} (o(1) + O_p(1))^2} \\
&= O_p(1)
\end{aligned}
\tag{82}
$$

The second equality is because

$$
\max_{t \leq T+1} |T^{-1/2} y_{i,t}(0)| \Rightarrow \max_{r \in [0,1]} \nu_i(r)
\tag{83}
$$

by the continuous mapping theorem.

To show Part (d), we combine (47) and (52), and have

$$
\|D_T(\hat{\beta}_i - \beta_i)\| = \left\| \begin{bmatrix} \hat{a}_i - a_i \\ \sqrt{T}(\hat{b}_i - b_i) \end{bmatrix} \right\| = o_p(1).
\tag{84}
$$

Therefore,

$$
\|(\hat{\theta} - \theta_0) D_T\|_F = \sqrt{\sum_{i=1}^{N} \|D_T(\hat{\beta}_i - \beta_i)\|^2} = o_p(1).
\tag{85}
$$

The second half of Part (d) is also satisfied since $\hat{\theta}^{(t)} = \hat{\theta}$ for each t.

Part (e) is assumed and Part (f) is trivial for $W_T = I$. $\qquad\square$

Proof of Theorem 3. We use similar strategy as we do in the proof of Theorem 2. Let

$$
\begin{aligned}
L_{1,T}(\epsilon) &= \left\{ \|(\hat{\theta} - \theta_0)\|_F \leq \epsilon, \max_{t=1,\dots,T} \|(\hat{\theta}^{(t)} - \theta_0)\|_F \leq \epsilon \right\}, \\
L_{2,T}(c) &= \left\{ \max_{t \leq T+1} \|D_T^{-1} x_t\| \leq c \right\}, \\
L_{3,T}(\eta) &= \left\{ \|\hat{G}' C' W_T C \hat{G} - G' C' W C G\|_F < \eta \right\}.
\end{aligned}
\tag{86}
$$

By Assumption 3(d), there exists a positive sequence $\{\epsilon_T\}_{T \geq 1}$ such that $\epsilon_T \to 0$ and $\Pr(L_{1,T}(\epsilon_T)) \to 1$. Let $c_T = 1/\sqrt{\epsilon_T}$. So we have $c_T \to \infty$ and $c_T \epsilon_T \to 0$. By Assumption 2(c), we must have $\Pr(L_{2,T}(c_T)) \to 1$. By Assumption 1(c) and Assumption 2(f), there exists a positive sequence $\{\eta_T\}_{T \geq 1}$ such that $\eta_T \to 0$ and $\Pr(L_{3,T}(\eta_T)) \to 1$. Let $L_T = L_{1,T}(\epsilon_T) \cap L_{2,T}(c_T) \cap L_{3,T}(\eta_T)$, then we have $\Pr(L_T) \to 1$ and $\Pr(L_T^c) \to 0$.

Suppose L_T holds. Then, for some $\theta = \hat{\theta}$ or $\theta = \hat{\theta}^{(t)}$ and for some $t = 1, \ldots, T$, we have

$$|\hat{P}_t(\theta) - P_t(\theta_0)| \leq |\hat{P}_t(\theta) - P_t(\theta)| + |P_t(\theta) - P_t(\theta_0)|. \tag{87}$$

Note that

$$
\begin{aligned}
|\hat{P}_t(\theta) - P_t(\theta)| &= \left| (Y_t - \theta x_t)'(\hat{G}'C'W_T C\hat{G} - G'C'WCG)(Y_t - \theta x_t) \right| \\
&\leq \|Y_t - \theta x_t\|^2 \|(\hat{G}'C'W_T C\hat{G} - G'C'WCG\|_F \\
&\leq \|u_t + (\theta_0 - \theta)x_t\|^2 \cdot \eta_T \\
&\leq (\|u_t\| + \|(\theta_0 - \theta)D_T D_T^{-1} x_t\|)^2 \eta_T \\
&\leq (\|u_t\| + \|(\theta_0 - \theta)D_T\|_F \|D_T^{-1} x_t\|)^2 \eta_T \\
&\leq (\|u_t\| + \epsilon_T c_T)^2 \eta_T \tag{88}
\end{aligned}
$$

and

$$
\begin{aligned}
|P_t(\theta) - P_t(\theta_0)| &= |(Y_t - \theta x_t)'G'C'WCG(Y_t - \theta x_t) - (Y_t - \theta_0 x_t)'G'C'WCG(Y_t - \theta_0 x_t)| \\
&\leq |(Y_t - \theta x_t)'G'C'WCG(Y_t - \theta x_t) - (Y_t - \theta x_t)'G'C'WCG(Y_t - \theta_0 x_t)| \\
&\quad + |(Y_t - \theta x_t)'G'C'WCG(Y_t - \theta_0 x_t) - (Y_t - \theta_0 x_t)'G'C'WCG(Y_t - \theta_0 x_t)| \\
&= |(u_t + (\theta_0 - \theta)x_t)'G'C'WCG(\theta_0 - \theta)x_t| + |((\theta_0 - \theta)x_t)'G'C'WCG u_t| \\
&\leq \|u_t + (\theta_0 - \theta)D_T D_T^{-1} x_t\| \|G'C'WCG\|_F \|(\theta_0 - \theta)D_T D_T^{-1} x_t\| \\
&\quad + \|(\theta_0 - \theta)D_T D_T^{-1} x_t\| \|G'C'WCG\|_F \|u_t\| \\
&\leq (\|u_t\| + \epsilon_T c_T)\|G'C'WCG\|_F \epsilon_T c_t + \epsilon_T c_T \|G'C'WCG\|_F \|u_t\| \\
&= (2\|u_t\| + \epsilon_T c_T)\|G'C'WCG\|_F \epsilon_T c_t. \tag{89}
\end{aligned}
$$

Combining (87), (88), and (89), we have

$$|\hat{P}_t(\theta) - P_t(\theta_0)| \leq g(\epsilon_T, c_T, \eta_T), \tag{90}$$

where

$$g_t(\epsilon_T, c_T, \eta_T) = (\|u_t\| + \epsilon_T c_T)^2 \eta_T + (2\|u_t\| + \epsilon_T c_T)\|G'C'WCG\|_F \epsilon_T c_t. \tag{91}$$

By Assumption 1(a), $g_t(\epsilon_T, c_T, \eta_T)$ is identically distributed across t for a fixed T.

To show part (a), note that under null,

$$P = (C\hat{\alpha} - d)'W_T(C\hat{\alpha} - d)$$

$$= (C(\alpha + Gu_{T+1} + o_p(1)) - d)'(W + o_p(1))(C(\alpha + Gu_{T+1} + o_p(1)) - d)$$

$$= (CGu_{T+1} + o_p(1))'(W + o_p(1))(CGu_{T+1} + o_p(1))$$

$$= u'_{T+1}G'C'WCGu_{T+1} + o_p(1). \tag{92}$$

The second equality is by Theorem 1. Since $P_\infty = u'_1 G'C'WCGu_1$, we have $P \to_d P_\infty$ by stationary of $\{u_t\}_{t \geq 1}$.

Part (b)-(d) can be shown using the same strategy as in the proof of Theorem 2, with $g_t(\epsilon_T, c_T, \eta_T)$ in place of $g_t(\epsilon_T, c_T)$, and θ in place of β, so is omitted here. $\qquad \square$

References

A. Abadie and M. D. Cattaneo. Econometric Methods for Program Evaluation. *Annual Review of Economics*, 10(1):465–503, 8 2018.

A. Abadie and J. Gardeazabal. The Economic Costs of Conflict: A Case Study of the Basque Country. *American Economic Review*, 93(1):113–132, 2 2003.

A. Abadie, A. Diamond, and J. Hainmueller. Synthetic Control Methods for Comparative Case Studies: Estimating the Effect of Californias Tobacco Control Program. *Journal of the American Statistical Association*, 105(490):493–505, 6 2010.

D. W. K. Andrews. End-of-Sample Instability Tests. *Econometrica*, 71(6):1661–1694, 2003.

D. W. K. Andrews and J. Kim. Tests for Cointegration Breakdown Over a Short Time Period. *Journal of Business & Economic Statistics*, 24(4):379–394, 10 2006.

G. Basse, A. Feller, and P. Toulis. Conditional randomization tests of causal effects with interference between units. 9 2017. URL http://arxiv.org/abs/1709.08036.

V. Chernozhukov, K. Wuthrich, and Y. Zhu. An Exact and Robust Conformal Inference Method for Counterfactual and Synthetic Controls. 12 2017.

T. G. Conley and C. R. Taber. Inference with Difference in Differences with a Small Number of Policy Changes. *Review of Economics and Statistics*, 93(1):113–125, 2 2011.

N. Doudchenko and G. W. Imbens. Balancing, Regression, Difference-In-Differences and Synthetic Control Methods: A Synthesis. 10 2016.

Z. Fang and A. Santos. Inference on Directionally Differentiable Functions. *The Review of Economic Studies*, 86(1):377–412, 9 2018.

B. Ferman and C. Pinto. Revisiting the Synthetic Control Estimator. 9 2016.

B. Ferman and C. Pinto. Placebo Tests for Synthetic Controls. 4 2017.

B. Ferman and C. Pinto. Inference in Differences-in-Differences with Few Treated Groups and Heteroskedasticity. *The Review of Economics and Statistics*, 7 2018.

J. Hahn and R. Shi. Synthetic Control and Inference. *Econometrics*, 5(4):52, 11 2017.

J. D. Hamilton. *Time series analysis*. Princeton University Press, 1994.

F. Hayashi. *Econometrics*. Princeton University Press, 2000.

K. T. Li. Statistical Inference for Average Treatment Effects Estimated by Synthetic Control Methods, 2017.

W. K. Newey and D. McFadden. Chapter 36 Large sample estimation and hypothesis testing. *Handbook of Econometrics*, 4:2111–2245, 1 1994.

G. Vazquez-Bare. Identification and Estimation of Spillover Effects in Randomized Experiments. 11 2017. URL `http://arxiv.org/abs/1711.02745`.

E. H. Zarantonello. Projections on Convex Sets in Hilbert Space and Spectral Theory. In *Contributions to Nonlinear Functional Analysis*, pages 237–424. Elsevier, 1971.